NP-hardness of the Clar number in general plane graphs

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Abstract

We prove that calculating the Clar number in general plane graphs is NP-hard.

1 Problem Definition

Let $G = (V, E)$ denote a 2-connected planar graph which has a perfect matching. For a planar embedding of G and a perfect matching M of G let F_M denote the set of those faces which alternate with respect to M. Note that faces in F_M are even. A pairwise vertex disjoint subset of F_M is a **Clar set with respect to** M. A subset C of the faces is a **Clar set** if there exists a perfect matching M for which C is a Clar set with respect to M. Note that a set of pairwise vertex disjoint even faces is a Clar set if and only if deleting all (the nodes of) these even faces the remaining graph still has a perfect matching. The **Clar number** of G, denoted by $Cl(G)$ is the maximum size of a Clar set.

It was proved by Abeledo and Atkinson [1] that the Clar number can be computed in polynomial time if G is bipartite planar.

In this quick proof we show that the general problem is NP-hard.

Theorem 1. *It is NP-hard to calculate the Clar number of a planar graph.*

2 Independent Set Problem

Definition 2. Let $\alpha(G)$ denote the maximum size of an independent set in G.

Lemma 3. *The Independent Set problem is NP-hard even for planar graphs with odd faces only.*

Proof. The independent set problem is NP-hard for planar graphs (see Problem GT20 in [2]). Let $G = (V, E)$ denote an instance of this problem. If G has an even face F, let G_F denote the planar graph attained from G by the following operation. We add three vertices a, b, c inside F and edges ab, bc, ca, au, bu, bv where u and v form an edge of F (see Figure 1).

Claim 4. $\alpha(G_F) = 1 + \alpha(G)$.

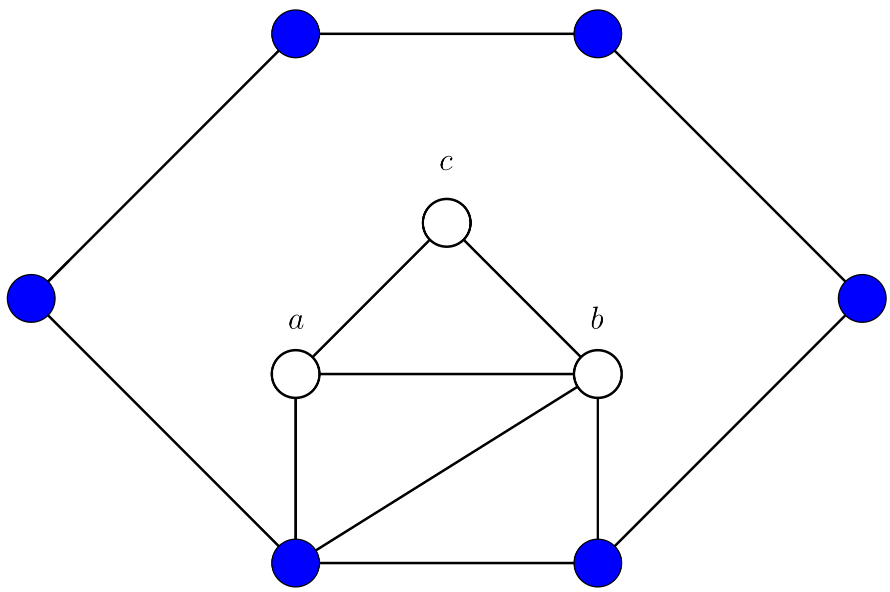

Figure 1: Eliminating even faces.

Proof. First, for an independent set I of G, clearly $I \cup c$ is independent in G_F and hence $\alpha(G_F) \geq 1 + \alpha(G)$. Second, an independent set I_F in G_F can contain at most one vertex from the set $\{a, b, c\}$. Since $I_F \setminus \{a, b, c\}$ is independent in G we get that $\alpha(G) \geq \alpha(G_F) - 1$. □

Note that the number of even faces of G_F is one less than that of G. Let $\mathbb{F}$ denote the set of even faces of G. By consecutively applying the above operation on every member of $\mathbb{F}$ we get another graph $G_{\mathbb{F}}$ for which $\alpha(G_{\mathbb{F}}) = \alpha(G) + |\mathbb{F}|$ and which has odd faces only. □

3 Proof of Theorem

Proof of Theorem 1. We prove the theorem by reducing the Independent Set problem for planar graphs with odd faces to the Clar number problem. Let $G = (V, E)$ denote a 2-connected instance of this problem (the proof can be easily extended to general connected graphs). We construct graph G' the following way: for every edge of G we add two vertices to G'. Let $uv \in E$ be an edge of G and let F_1 and F_2 denote the faces uv is incident to. We add vertices uv_{F_1} and uv_{F_2} to G' along with the edge $uv_{F_1}uv_{F_2}$. If edges uv and vw are neighboring edges on a face F, then we add edge uv_Fvw_F to G'. It is easy to see that G' is planar (see Figure 2). The faces of G' correspond to the faces and vertices of G, and if G has odd faces only, then all the even faces of G' are the ones corresponding to vertices of G. Note that G' trivially has a perfect matching M consisting of the edges of the form $uv_{F_1}uv_{F_2}$, for every $uv \in E$. Since M is alternating on every even face of G', corresponding to a vertice of G, that is, on every even face of G', for this graph the Clar number equals the maximum size of a Clar set with respect to M. The Clar sets of G' and the independent sets of G have

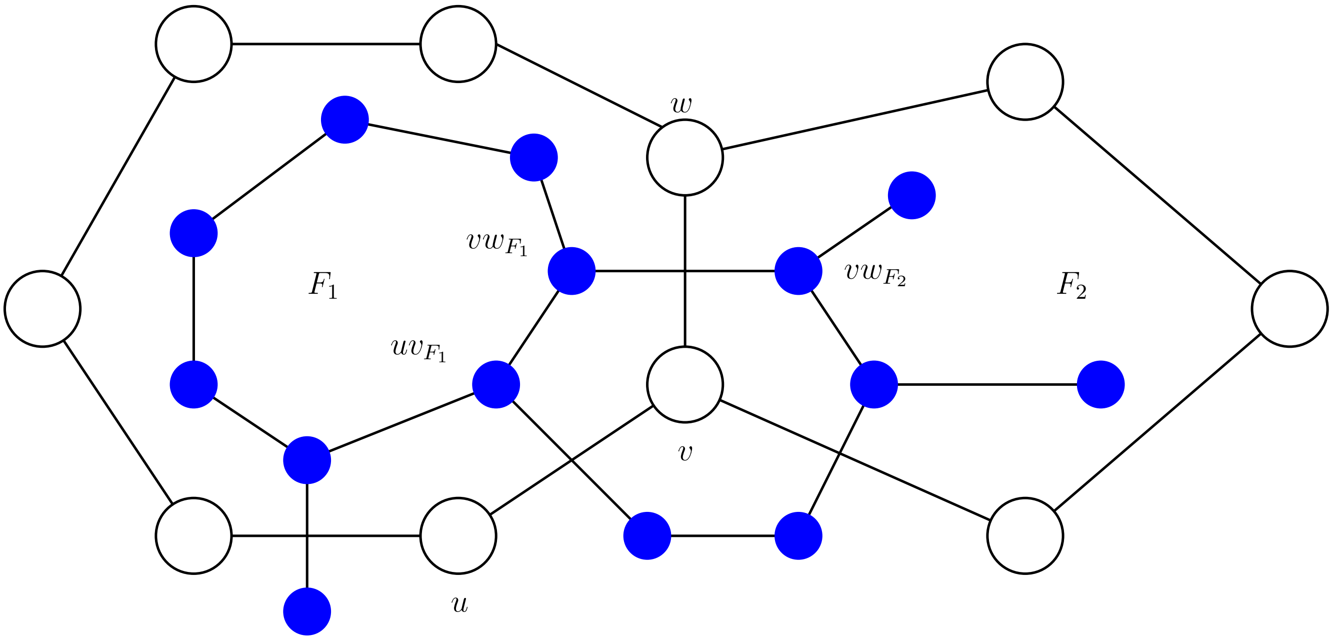

Figure 2: Reduction of Independent Set

a one to one correspondance, proving the theorem.

□

Corollary 5. *It is also NP-hard to find a maximum cardinality Clar set with respect to a fixed perfect matching.*

4 Open questions

We proved the NP-hardness of the Clar number for general planar graphs. The classical case of the Clar number, when G has exactly twelve pentagonal faces and every other face is a hexagon, however, is still open. If we were able to specialize the Independent Set problem further to 3-regular planar graphs with odd faces, we would get that the Clar number is NP-hard for graphs whith only hexagonal even faces.

References

[1] H. G. Abeledo and G. W. Atkinson, *A min-max theorem for plane bipartite graphs*, Discrete Applied Mathematics **158** (2010), no. 5, 375–378.

[2] M. R. Garey and D. S. Johnson, *Computers and intractability: A guide to the theory of NP-completeness*, W. H. Freeman & Co., New York, NY, USA, 1979.