A note on conservative costs

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Abstract

Let $G = (V, E)$ be an undirected graph and $c : E \to \{-1, +1\}$ a conservative cost function. We show that the problem of determining the maximum number of edges whose cost can be changed from 1 to -1 without violating the conservativeness of c is NP-complete. A similar result about directed graphs is also proved.

1 Introduction

Let $G = (V, E)$ be an undirected graph and $c : E \to \{-1, +1\}$ a cost function. For a cycle C, its node-set and edge-set is denoted by V_C and E_C, respectively. If $D = (V, A)$ is a directed graph and C is a directed cycle, then A_C is used instead of E_C. We call c *conservative* if $c(E_C) = \sum_{e \in E_C} c_e \geq 0$ for each cycle C of G. In directed graphs this is required only for directed cycles. A subset of edges $F \subseteq E$ (or arcs, in the directed case) all having cost 1 is called *negatable* if changing their cost to -1 results in another conservative cost function. We denote the cost function thus obtained by c_F. A natural question is to determine the maximum size of a negatable set. We will show that this problem is NP-complete both in the undirected and directed case.

Throughout we use the following notation. The set of edges incident to a node $v \in V$ is denoted by $\delta(v)$. Given a directed graph D and a node $v \in V$, $\delta^-(v)$ and $\delta^+(v)$ stand for the set of arcs entering and leaving v, respectively. Each notation is used with subscripts when working with different graphs simultaneously.

2 The undirected case

Let $G = (V, E)$ be an undirected graph. The *distance* $d(u, v)$ of two nodes $u, v \in V$ is the length of the shortest path between them. We call a subset of nodes $S \subseteq V$ *k-stable* if $d(u, v) \geq k + 1$ for each $u, v \in S$. That is, 1-stable sets are the usual stable sets.

Our proof for the undirected case is based on the following complexity result.

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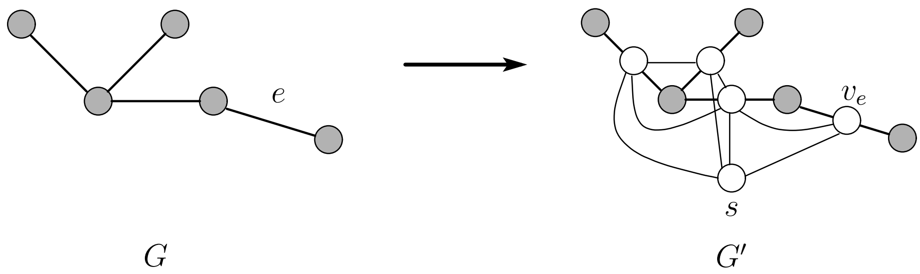

Figure 1: Construction of G'

Theorem 1. *The problem of determining the maximum size of a 2-stable set is NP-complete.*

Proof. Split each edge e into two parts with a new node v_e. For a node $v \in V$, add the edges $v_e v_f$ to the graph for each $e, f \in \delta(v)$. Finally, add a 'super node' s to the node-set and edges sv_e for each $e \in E$ (see Figure 1). Let $G' = (V', E')$ be the graph thus arising. By abuse of notation, a node $v \in V$ and its pair in V' is identified, that is, $V \subseteq V'$ is assumed.

Let S be a stable set in the original graph G. Then S clearly forms a 2-stable set in G'.

Now let S' be a 2-stable set in G'. If $s \in S'$ then $|S'| = 1$ and any node $v \in V$ forms a stable set S with size 1. Assume that $s \notin S'$. There is at most one edge $e \in E$ with $v_e \in S'$ because of the edges leaving s. If there is no such node then S' forms a stable set in G. If $v_e \in S'$ for some edge $e = uv$ then we claim that both $S' - v_e + u$ and $S' - v_e + v$ are stable sets in G. To see this, we only have to show that S' does not contain a neighbour of u or v. Indeed, this follows from the fact that such a node has a distance at most 2 from v_e in G' and S' supposed to be 2-stable.

It follows from the above that the maximum size of a stable set in G is equal to the maximum size of a 2-stable set in G'. As determining the maximum size of a stable set is a well-known NP-complete problem, the theorem follows. $\square$

Now we turn to our main result.

Theorem 2. *The problem of determining the maximum size of a negatable set in an undirected graph is NP-complete.*

Proof. Let $G^\circ = (V^\circ, E^\circ)$ be the graph obtained from G as follows. For each node $v \in V$ we add the nodes v', v'', and for each edge $e \in E$ we add a node v_e to V°. We also add a 'super node' s to the new graph. For an edge $e = uv \in E$, we add edges $u'v_e, u''v_e, v'v_e$ and $v''v_e$ to E°. Let also $sv', v'v'' \in E^\circ$ for each $v \in V$ (see Figure 2). Define a cost function on E° as

$$c(e) = \begin{cases} -1 & \text{if } e \text{ is of form } v'v'' \text{ for some } v \in V, \\ 1 & \text{otherwise.} \end{cases}$$

It is easy to check that c is a conservative cost function. Also, if the cost of an edge $e \in E^\circ$ is negatable then $e \in \delta(s)$.

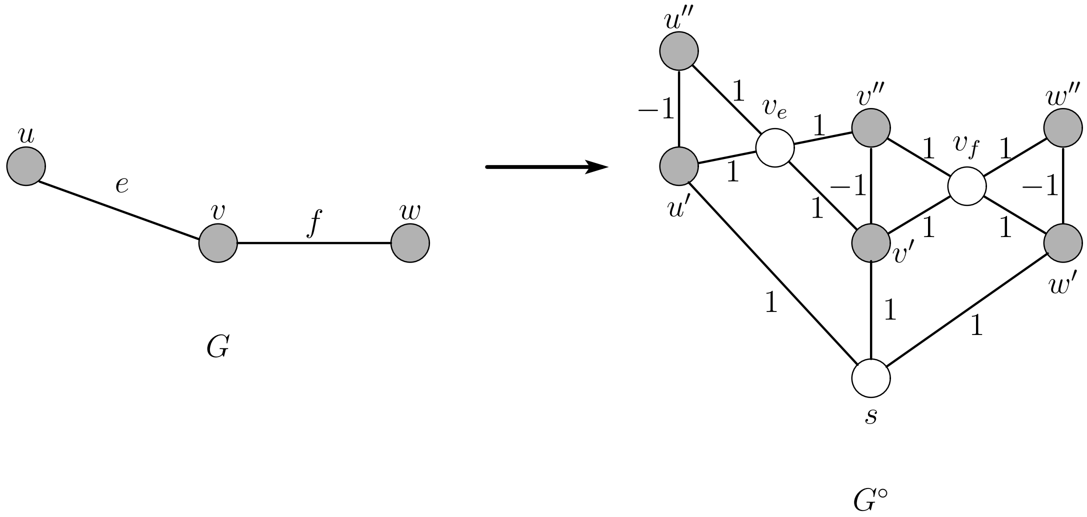

Figure 2: Construction of G°

Let S be a 2-stable set in G and $F = \{sv' \in E^\circ : \ v \in S\}$. We claim that F is negatable. To see this, it suffices to show that $c_F(E^\circ_C) \geq 0$ for each cycle C that contains s. Also, we may assume that the cycle does not contain a pair of edges $v'v_e, v''v_e$ for some $v \in V$ and $e \in \delta(v)$ as replacing them with edge $v'v''$ would decrease the total cost of the cycle.

Let $\{v_1, ..., v_k\}$ denote those nodes around the cycle that are different from s and are not of form v_e for some $e \in E$. We may assume that s lies between v_1 and v_k on C. Then we have

$$c_F(E^\circ_C) \geq -1 - \chi(d_G(v_1, v_k) \leq 2) - k + 2(k-1) \tag{1}$$

where

$$\chi(d_G(u,v) \leq 2) = \begin{cases} -1 & \text{if } d(u,v) \geq 3 \text{ in } G, \\ 1 & \text{otherwise.} \end{cases}$$

Indeed, $-1 - \chi(d_G(v_1, v_k) \leq 2)$ is a lower bound for the cost of edges sv_1, sv_k as S was 2-stable, while the cycle may use at most k edges of form $v'_i v''_i$, and $2(k-1)$ is the cost of the steps when the cycle goes from v'_i or v''_i to v'_{i+1} or v''_{i+1}. Clearly, $c_F(E^\circ_C) \geq 0$ for $k \geq 4$. If $k = 2$ or 3 then $d_G(v_1, v_k) \leq 2$, hence $c_F(E_C) \geq 0$ follows.

Now let $F \subseteq E^\circ$ be a negatable set and $S = \{v \in V : \ sv' \in F\}$. Then S is 2-stable. Indeed, if $u, v \in S$ and $e = uv \in E$ then $c_F(\{su', u'u'', u''v_e, v_ev'', v''v'\}) = -2$, contradicting the negatability of F. If $u, w \in S$ and there is a $v \in V$ with $e = uv, f = vw \in E$ then $c_F(\{su', u'u'', u''v_e, v_ev', v'v'', v''v_f, v_fw'', w''w'\}) = -1$, contradicting the negatability of F, again.

It follows from the above that the maximum size of a 2-stable set in G is equal to the maximum size of a negatable set in G°. As determining the maximum size of a 2-stable set is NP-complete by Theorem 1, the theorem follows. $\square$

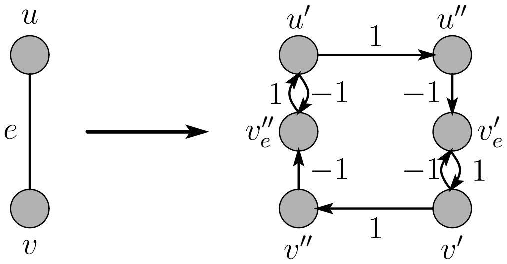

Figure 3: Construction of D

3 The directed case

Let $D = (V, A)$ be a digraph and $c : A \to \{-1, +1\}$ a conservative cost function.

Theorem 3. *The problem of determining the maximum size of a negatable set in a directed graph is NP-complete.*

Proof. Let $G = (V, E)$ be an undirected graph. Let $D = (W, A)$ be the digraph obtained from G by replacing each node $v \in V$ by an arc $v'v''$ and each edge $e = uv \in E$ by arcs $u''v'_e, v'_ev', v'v'_e, v''v''_e, v''_eu'$ and $u'v''_e$ (see Figure 3).

Define a cost function on A as

$$c(a) = \begin{cases} -1 & \text{if the head of } a \text{ is } v'_e \text{ or } v''_e \text{ for some } e \in E, \\ 1 & \text{otherwise.} \end{cases}$$

It is easy to see that c is conservative. Also, if $a \in A$ is negatable then it is an arc corresponding to a node of V, that is, $a = v'v''$ for some $v \in V$.

Let $F \subseteq A$ be a negatable set. Then $S = \{v \in V : v'v'' \in F\}$ is a stable set in G. Indeed, if $u, v \in S$ and $e = uv \in E$, then $c_F(\{u'u'', u''v'_e, v'_ev', v'v'', v''v''_e, v''_eu'\}) = -2$, a contradiction.

Now take a stable set $S \subseteq V$ of G. We claim that $F = \{v'v'' : v \in S\}$ is a negatable set. Let C be a directed cycle in D. Then either $A_C = \{v'v'_e, v'_ev'\}$ or $\{v'v''_e, v''_ev'\}$ for some $v \in V$ and $e \in \delta(v)$, or $V_C = \{v'_1, v''_1, x_1, v'_2, v''_2, x_2, ..., x_{k-1}, v'_k, v''_k\}$ where $x_i = v'_{e_i}$ or v''_{e_i} and $e_i = v_iv_{i+1}$. It is easy to see that, as S is stable, $c_F(A_C) \geq 0$.

The above implies that the maximum size of a stable set in G is equal to the maximum size of a negatable set in D. As determining the maximum size of a stable set is NP-complete, the theorem follows.

□