Constructive characterization of dumpy digraphs

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Abstract

We give a constructive characterization of 2-dumpy digraphs.

1 Introduction

Let $D = (V + r, A)$ be a digraph with root r. D is called *k-dumpy* if the root has in-degree 0, every non-root node has in-degree k and every non-singleton set not containing the root has in-degree at least $k + 1$. The motivation of working with dumpy graphs is the following. When examining problems concerning rooted directed graphs with $\varrho(X) \geq k$ for each $X \subseteq V$, a set $Z \subseteq V$ with $\varrho(Z) = k$ often can be contracted as to get a smaller digraph to work on and apply induction.

Also, the $k = 2$ case has a motivation from rigidity. Let $G = (V, E)$ be a two-dimensional minimally rigid graph. By adding an extra node r to G and connecting it to nodes x and y with one and two edges, respectively, one gets a graph that has a rooted 2-edge connected orientation from r (this can be proved easily using that G is a $(2, 3)$-graph). Such an orientation results in a 2-dumpy digraph. Hence the constructive characterization of dumpy digraphs for the case $k = 2$ gives a new proof of the Henneberg construction. Not surprisingly, the proof relies on the same ideas as the one for the undirected case.

Throughout we use the following notation. Given a directed graph $D = (V + r, A)$ with root r, $\varrho_D(X) = \varrho_A(X) = \varrho(X)$ and $\delta_D(X) = \delta_A(X) = \delta(X)$ denote the number of edges entering and leaving X, respectively. Often we do not distinguish between a one-element set and its only element. For example, the in-degree $\varrho(\{v\})$ of a singleton $\{v\}$ is abbreviated by $\varrho(v)$. For disjoint sets $X, Y \subseteq V + r$ the number of arcs going from X to Y is denoted by $\overrightarrow{d}(X, Y)$. The complement of a set X is denoted by $\overline{X}$. In a k-dumpy digraph a set $X \subseteq V$ is called *tight* if $\varrho(X) = k + 1$. Let $e = uv$ and $f = vw$ be two arcs of D. *Splitting off* the pair $\{e, f\}$ means that we replace the two arcs e, f by a new one $g = uw$.

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2 Characterization of 2-dumpy digraphs

Our main result is the following theorem.

Theorem 1. *A digraph $D = (V + r, A)$ is 2-dumpy if and only if it can be built up from a single (root) node r using the following three steps:*

(i) adding a new node v and two edges entering v that can be parallel only if their tail is r;

(ii) subdividing an edge xy with a new node v and adding a new edge uv for which $u \neq y$ and if $x = u$ then it is the root node r;

(iii) redirecting a directed cycle.

In addition, the building steps can be chosen so that no two Steps (iii) are used consecutively.

Proof. It is easy to see that the building steps preserve dumpiness. Note that since reversing a dicycle preserves the in-degree of every set, it preserves dumpiness too.

To prove the other direction, we show that either there is a node for which the inverse of Steps (i) or (ii) can be applied, or we can redirect a dicycle to get such a node. That would clearly prove the theorem.

If there is a node v with out-degree 0, then $G - v$ is dumpy, too. If there is no such node then there is at least one with out-degree one. Reversing Step (*ii*) on such a node means splitting off one of the entering edges with the outgoing one and deleting the node itself.

Lemma 2. *There is a node v with out-degree one such that there is a directed cycle through v.*

Proof. Take the strongly connected components of D and consider their topological order. Take a component C that corresponds to a sink in this order. As we assumed that there is no node with out-degree 0 we have $|C| \geq 2$. Also, $\varrho(v) = 2$ for each $v \in V$ and $\varrho(C) \geq 1$ (in fact $\varrho(C) \geq 3$ by the dumpiness of D), so there is a node $v \in C$ with $\delta(v) = 1$. C is strongly connected and $|C| \geq 2$ so there is a directed cycle through v in C. □

Lemma 3. *Let v be a node with out-degree one such that there is a directed cycle through v. Then there is a splittable pair at v, or there is one after reorienting an arbitrary dicycle containing v.*

Proof. Let x and y denote the tails of the edges entering v while let z be the head of the edge leaving v. Assume first that $x = r$. Then the pair $\{rv, vz\}$ is splittable as such a step preserves the in-degree of every set $X \subseteq V - v$.

So assume that $x, y \neq r$. Note that x, y and z are pairwise different due to the dumpiness of D. The pair $\{xv, vz\}$ cannot be splitted off if and only if there is a tight set X (that is, $\varrho(X) = 3$) containing x and z but not v. For such a tight set we say that it *covers* (x, z).

Claim 4. *If X and Y are tight sets for which $|X \cap Y| \geq 2$, then $X \cup Y$ is tight. If X_1, X_2, X_3 are tight sets for which $X_1 \cap X_2 \cap X_3 = \emptyset$ and $|X_i \cap X_j| = 1$ for $1 \leq i < j \leq 3$, then $X_1 \cup X_2 \cup X_3$ is tight.*

Proof. The first part of the claim easily follows from the submodularity of the in-degree function and the dumpiness of D.

Let $x_{ij} = X_i \cap X_j$ for $1 \leq i < j \leq 3$, $U = X_1 \cup X_2 \cup X_3$ and $X = \{x_{12}, x_{13}, x_{23}\}$. Then we have

$$\begin{aligned} \varrho(U) &= \overrightarrow{d}(\overline{U}, U - X) + \overrightarrow{d}(\overline{U}, X), \\ 6 &= \varrho(x_{12}) + \varrho(x_{13}) + \varrho(x_{23}) = \overrightarrow{d}(\overline{U}, X) + \overrightarrow{d}(U - X, X) + i(X). \end{aligned}$$

On the other hand, a simple computation shows that

$$\begin{aligned} 9 &= \varrho(X_1) + \varrho(X_2) + \varrho(X_3) \\ &\geq 2\overrightarrow{d}(\overline{U}, X) + \overrightarrow{d}(U - X, X) + \overrightarrow{d}(\overline{U}, U - X) + i(X) \\ &= 6 + \varrho(U). \end{aligned}$$

The last inequality implies $3 \geq \varrho(U)$, that is, $X_1 \cup X_2 \cup X_3$ is tight. □

From Claim 4 the following follows.

Claim 5. *There are no tight sets covering (x, z), (y, z) and (x, y).*

Proof. Assume that there are three tight sets X, Y and Z covering (x, z), (y, z) and (x, y), respectively. If $|X \cap Y| \geq 2$, then $X \cup Y$ is also tight by Claim 4. But then $\varrho(X \cup Y \cup \{v\}) = 2$, contradicting the dumpiness of D. So $X \cap Y = \{z\}$.

It can be proved similarly that $X \cap Z = \{x\}$ and $Y \cap Z = \{y\}$. Claim 4 implies that $X \cup Y \cup Z$ is tight. But then $\varrho(X \cup Y \cup Z \cup \{v\}) = 2$, contradicting the dumpiness of D. □

By Claim 5, at least on of the pairs (x, y), (x, z) and (y, z) is not covered by a tight set. If (x, z) or (y, z) is such a pair, then the pair $\{xv, vz\}$ or $\{yv, vz\}$ can be splitted off, respectively.

If the only such pair is (x, y) then take any directed cycle that contains v and reorient its edges. Clearly, the digraph remains dumpy and one of the pairs $\{xv, vy\}$ or $\{yv, vx\}$ becomes splittable after the reorientation as (x, y) was not covered by tight sets. □

The last part of the theorem immediately follows from the above. Indeed, if none of the reverse of Steps (i) and (ii) can be used then we can reorient a cycle that provides a node for what Step (ii) can be applied. □

3 Necessity of Step (iii)

We give an example that shows the necessity of Step (iii) for constructing 2-dumpy digraphs.

Let D be the digraph shown on Figure 1. It is easy to check that D is 2-dumpy. There are three nodes with out-degree 1 denoted by x, y and z. However, there is no splittable pair at these nodes because of the tight sets (for example, at node x, shown with dotted lines), hence Step (iii) should be applied first.

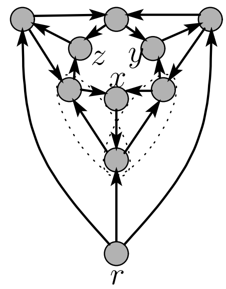

Figure 1: Necessity of Step (iii)

A similar construction shows that the reorientation of directed cycles is also needed when constructing k-dumpy graphs for larger values of k. Let $D = (V, A)$ be the following digraph (see Figure 2):

$$V = \{u_0, ..., u_k, v_0, ..., v_k, w_1, ..., w_k, r\} \text{ and}$$
$$A = \bigcup_{i=0}^{k} \{rw_i, v_iu_i, v_{i+1}u_i, u_iw_i, w_iv_{i+1}, (k-2)\times w_iu_i, (k-1)\times w_iv_i, (k-1)\times v_{i+1}w_i\}$$

where indices are meant modulo $k+1$ and multipliers $(k-2)$ and $(k-1)$ denote parallel arcs. The digraph thus arising is k-dumpy and the nodes with out-degree one are $u_0, ..., u_k$. However, there is no splittable pair at these nodes because of the tight sets of form $\{v_i, w_i\}$ and $\{v_{i+1}, w_i\}$.

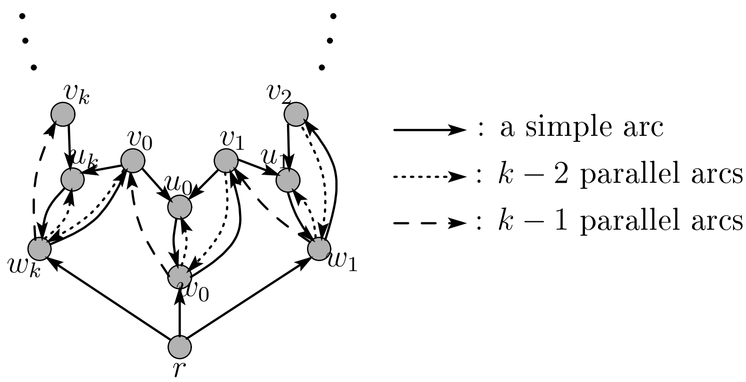

Figure 2: Necessity of Step (iii) for $k \geq 3$