Subgraphs with restricted degree differences

Kristóf Bérczi ⋆

Abstract

Given a graph $G = (V, E)$ and two functions $c_l, c_u : V \to \mathbb{Z}$ with $c_l \le c_u$, we give a characterization of the existence of two subgraphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ such that $c_l(v) \le d_1(v) - d_2(v) \le c_u(v)$ for each $v \in V$.

1 Introduction

For an undirected graph $G = (V, E)$, the sets of edges induced by and having exactly one end in $X \subseteq V$ are denoted by $E[X]$ and $\delta(X)$, respectively. For disjoint subsets X, Y of V, $E[X, Y]$ denotes the set of edges between X and Y. We define $d(v) = |\delta(v)|$ where loops in G are counted twice. Also, $d(X, Y)$ stands for the number of edges going between disjoint subsets X and Y. For a node $v \in V$, we abbreviate the set $\{v\}$ by v. Sometimes we use these notations with subscripts when only a subset $F \subseteq E$ is considered, we work with different graphs simultaneously, or we have a vector on the edges $x \in \mathbb{Z}^E$. For example, $d_x(v) = \sum_{e\in\delta(v)} x(e)$ for a node $v \in V$. We use $f(Z) = \sum_{v\in Z} f(v)$ for a function $f : V \to \mathbb{Z}$ and a set $Z \subseteq V$.

The following problem was proposed by Frank [1].

Problem 1. Given a graph $G = (V, E)$, give a characterization of the existence of a 'larger' and a 'smaller' subgraph $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$, respectively, such that

$$d_1(v) = d_2(v) + 1 \text{ for all } v \in V. \tag{1}$$

We give a slight generalization of this problem using upper and lower bounds.

Problem 2. Given a graph $G = (V, E)$, lower and upper bounds $c_l, c_u : V \to \mathbb{Z}$ with $c_l \le c_u$, give a characterization of the existence of two subgraphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ such that

$$c_l(v) \le d_{G_1}(v) - d_{G_2}(v) \le c_u(v) \text{ for all } v \in V. \tag{2}$$

Clearly, with choice $c_l, c_u \equiv 1$ we get Problem 1.

⋆MTA-ELTE Egerváry Research Group (EGRES), Department of Operations Research, Eötvös University, Pázmány P. s. 1/c, Budapest, Hungary, H-1117. Supported by the Hungarian National Foundation for Scientific Research (OTKA) grant CK80124. E-mail: `berkri@cs.elte.hu`

2 Characterization

Let $b_l(v) = d_G(v) + c_l(v)$ and $b_u(v) = d_G(v) + c_u(v)$. Our main observation is the following.

Theorem 3. *There exist G_1 and G_2 satisfying (2) if and only if there exist an $x \in \mathbb{Z}^E$ satisfying*

$$0 \le x(e) \le 2 \text{ for all } e \in E, \tag{3}$$

$$b_l(v) \le d_x(v) \le b_u(v) \text{ for all } v \in V. \tag{4}$$

Proof. First we prove necessity. Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ be subgraphs satisfying (2). We may assume that $E_1 \cap E_2 = \emptyset$ as common edges can be simply left out from both graphs. Define $x \in \mathbb{Z}^E$ as

$$x(e) = \begin{cases} 2 & \text{if } e \in E_1, \\ 0 & \text{if } e \in E_2, \\ 1 & \text{otherwise.} \end{cases}$$

Then x clearly satisfies (3) and we have

$$d_x(v) = 2d_1(v) + d(v) - (d_1(v) + d_2(v)) = d_1(v) - d_2(v) + d(v)$$

and $b_l(v) \le d_x(v) \le b_u(v)$ immediately follows from (2).

To see sufficiency, take an $x \in \mathbb{Z}^E$ satisfying (3) and (4). Define two subgraphs with

$$E_1 = \{e \in E : x(e) = 2\},$$
$$E_2 = \{e \in E : x(e) = 0\}.$$

Then we have

$$d_1(v) - d_2(v) = 2d_1(v) + d(v) - (d_1(v) + d_2(v)) = d_x(v),$$

and $c_l(v) \le d_{G_1}(v) - d_{G_2}(v) \le c_u(v)$ follows from (4). □

By the above theorem, we can characterize the existence of the required subgraphs. We will use the following theorem which can be derived from fundamental results of Tutte and Edmonds about matchings (see eg. [2]).

Theorem 4. *Let $G = (V, E)$ be a graph and let $b_l, b_u \in \mathbb{Z}^V$ with $b_l \le b_u$ and $p, q \in \mathbb{Z}^E$ with $p < q$. Then there exists an $x \in \mathbb{Z}^E$ satisfying*

$$p(e) \le x(e) \le q(e) \text{ for all } e \in E,$$
$$b_l(v) \le d_x(v) \le b_u(v) \text{ for all } v \in V$$

if and only if for each partition T, U, W of V, the number of components K of $G[T]$ with $b_l(K) = b_u(K)$ and $b_u(K) + p(E[K, W]) + q(E[K, U])$ odd is at most $b_u(U) - 2q(E[U]) - q(E[T, U]) - b_l(W) + 2p(E[W]) + p(E[T, W])$.

In our case, $p \equiv 0$, $q \equiv 2$, $b_l(v) = d_G(v) + c_l(v)$ and $b_u(v) = d_G(v) + c_u(v)$ for all $v \in V$. Hence for a subset $K \subseteq V$, $b_l(K) = c_l(K) + 2|E[K]| + d(K)$ and similarly for $b_u(K)$. By Theorem 4, we have the following.

Theorem 5. *Let $G = (V, E)$ be a graph and let $c_l, c_u : V \to \mathbb{Z}$ with $c_l \leq c_u$. There exist two subgraphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ such that*

$$c_l(v) \leq d_{G_1}(v) - d_{G_2}(v) \leq c_u(v) \text{ for all } v \in V \tag{5}$$

if and only if for each partition T, U, W of V, the number of components K of $G[T]$ with $c_l(K) = c_u(K)$ and $c_u(K) + d(K)$ odd is at most $c_u(U) - c_l(W) - 2|E[U]| - 2|E[W]| - |E[T, U \cup W]|$.

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References

[1] A. Frank, *Personal communication* (2007)

[2] A. Schrijver, *Combinatorial Optimization*, Chapter 35