A unified proof for Karzanov's exact matching theorem

Hans-Florian Geerdes* and Jácint Szabó**

Abstract

We give a short inductive proof for a pair of theorems of Karzanov characterizing when complete and complete bipartite graphs with red and blue edges have a perfect matching with exactly k red edges. In contrast with Karzanov's approach, our proof handles both cases simultaneously.

1 Introduction

Finding a perfect matching in a graph with edges colored red and blue, containing a specified number of red edges, is the **exact matching problem**, introduced by Papadimitriou and Yannakakis [6]. This problem admits an RP-algorithm (Mulmuley, Vazirani, Vazirani [5], see also Lovász [4]), but only some special graph classes are known for which it is polynomial time solvable. For graphs which can be embedded into a fixed orientable surface, Galluccio and Loebl [2] gave a pseudo-polynomial algorithm based on Pfaffian orientations, generalizing the analogous result of Barahona and Pulleyblank [1] on planar graphs. For complete bipartite graphs and complete graphs Karzanov [3] gave a characterization to the exact matching problem. This is rephrased in Theorem 4 in the present paper. Yi, Murty, and Spera [7] gave an alternative proof to Karzanov's characterization, and also the first polynomial algorithm to complete bipartite and complete graphs. These characterizations are the starting point of the present note, whose aim is to give a short and unified proof to Karzanov's theorem.

We remark that if the number of colors is not restricted to two, then the analogous problem is NP-complete. Indeed, the 3-dimensional perfect matching problem in 3-partite graphs can be reduced to the problem of finding a multicolored perfect matching in an n-edge-colored bipartite graph $K_{n,n}$.

*Zuse Institute, Berlin, Optimization Department, Takustr. 7, D-14195 Berlin, Germany. Research is supported by the DFG research center MATHEON "Mathematics for key technologies" in Berlin, e-mail: `geerdes@zib.de`

**Zuse Institute, Berlin, Optimization Department, Takustr. 7, D-14195 Berlin, Germany; and MTA-ELTE Egerváry Research Group (EGRES), Institute of Mathematics, Eötvös University, H-1117 Budapest, Pázmány P. s. 1/C, Hungary. The author received a grant (no. CK 80124) from the National Development Agency of Hungary, based on a source from the Research and Technology Innovation Fund. e-mail: `jacint@elte.hu`.

Figure 1: An example to coloring (c1) in $K_{4,4}$. Only the red edges are shown.

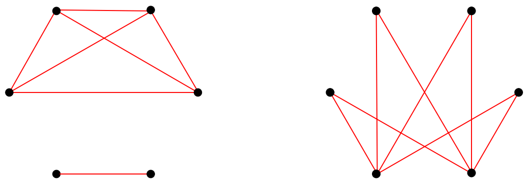

Figure 2: An example to coloring (c1) in K_6. Only the red edges are shown.

Throughout, $G = (V, E)$ is a simple undirected graph whose edge set is partitioned into red and blue edges: $E = E_R \dot\cup E_B$. Let $G_R = (V, E_R)$ and ν_R be the maximum size of a matching in G_R (similarly for G_B and ν_B). A subgraph with k red and ℓ blue edges is a (k,ℓ)**-subgraph**. We analogously define (odd, odd), (odd, 1), etc. subgraphs. An l**-circuit** is a circuit of length l. Complete and complete bipartite graphs are **full**, K_{2n} and $K_{n,n}$ are **balanced**. **Component** in a graph means connected component. If $v \in V$ is incident to no, say, red edge, then $\{v\}$ appears in G_R as a **singleton** component.

2 The proof

In the focus of the proof stand some special types of colorings.

- Coloring (c1): All components of G_R and G_B are full.
- Coloring (c2r): All components of G_R are balanced.
- Coloring (c2b): All components of G_B are balanced.

Some examples can be seen in Figures 1, 2 and 3. For x = 1, 2r or 2b we use the notation $G \sim$ (cx) if G has coloring of type (cx). It is easy to characterize how a coloring of type (c1) may look like. If G is a balanced bipartite graph with classes U and W and $G \sim$ (c1), then there exist partitions $U = U_1 \dot\cup U_2$ and $W = W_1 \dot\cup W_2$ (U_i, W_i can also be empty) such that G_R is exactly the union of K_{U_1,W_1} and K_{U_2,W_2}, and G_B is exactly the union of K_{U_1,W_2} and K_{U_2,W_1}. If $G = (V, E)$ is a balanced complete graph and $G \sim$ (c1), then there exists a partition $V = V_1 \dot\cup V_2$, where $|V_i| \geq 2$ and even for

Figure 3: An example to coloring (c2r) in $K_{5,5}$ and K_{10}. Only the red edges are shown.

$i = 1, 2$, such that one of G_R and G_B is K_{V_1,V_2}, while the other is the union of K_{V_1} and K_{V_2}.

The following lemma is easy to check by the help of Figures 1, 2 and 3.

Lemma 1. *Let G be balanced and $k_R \leq \nu_R$, $k_B \leq \nu_B$.*

- *If $G \sim$ (c1) and $k_B \equiv \nu_B \pmod 2$ (and then also $k_R \equiv \nu_R \pmod 2$),*
- *or if $G \sim$ (c2r) and $k_B \in \{0, 2\}$,*
- *or if $G \sim$ (c2b) and $k_R \in \{0, 2\}$,*

then G has a (k_R, k_B)-matching. Furthermore, every edge $f \in E$ belongs to a (k_R, k_B)-matching, except of course if f is red and $k_R = 0$, or if f is blue and $k_B = 0$. □

Lemma 2. *Let G be balanced.*

(l1) *Every 4-circuit is (even, even) $\iff G \sim$ (c1).*

(l2r) *G has a perfect red matching and has no 4-circuit which is (odd, 1) $\iff G \sim$ (c2r).*

(l2b) *G has a perfect blue matching and has no 4-circuit with is (1, odd) $\iff G \sim$ (c2b).*

Proof. We only focus on (l1) and (l2r), as (l2b) is analogous to (l2r). Directions $\Longleftarrow$ are clear in both cases. For directions $\Longrightarrow$, instead of a circuit, we first prove the existence of a **cycle**, that is a closed sequence of not necessarily disjoint or distinct edges, with the requirement on the number of colors (i.e., the cycle is (even, even) for (l1) and even length but not (odd, 1) for (c2r)). First, if a component of G_R induces both a blue edge and an odd red circuit, then it is not hard to construct an (odd, 1)-cycle. Thus we may assume that every component of G_R is either bipartite or complete. An (odd, 1)-circuit can be found in a non-full bipartite component, too, hence we may assume that every component of G_R is full.

(l1) In the non-bipartite case, if G_R has at least three components, or if G_R has two components at most one of which is a non-bipartite graph, then a $(1, 3)$-circuit can be created. In all remaining cases $G \sim$ (c1). In the bipartite case, if G_R has at least three non-singleton components, or if G_R contains two non-singleton and at least one singleton component, then a $(1, 3)$-circuit can be created. In all remaining cases $G \sim$ (c1).

(l2r) G has a perfect red matching, thus all components of G_R are balanced.

Next we show that if G has an (odd, odd)-cycle C, then G has a 4-circuit C' which is (odd, odd), moreover, if C was (odd, 1), then C' can be chosen to be (3,1). We denote the nodes of C by $w_1, \ldots, w_{2m}$ wrt. their ordering. Observe that if C has length at most 4, then because of the parity condition, it is already a 4-circuit and we are done. Otherwise we try to shortcut C at a node-pair w_i, w_{i+3}. If $w_i \neq w_{i+3}$, then $w_i w_{i+3}$

is indeed an edge and both $C_1 := w_i, \dots, w_{i+3}$ and $C_2 := w_{i+3} \dots w_i \pmod{2m}$ are shorter even cycles. Moreover, for either $j = 1$ or $j = 2$, C_j is (odd, odd), and if C is (odd, 1), then C_j is also (odd, 1). So we can apply induction.

Finally, we show that a node-pair w_i, w_{i+3} with $w_i \neq w_{i+3}$ can always be found. If C has length at least 8, then we may assume that w_0, w_3 or w_3, w_6 would do, because otherwise $w_0 = w_6$ and so either $w_0, \dots, w_6$ or $w_6 \dots w_0 \pmod{2m}$ are shorter cycles with the required parity condition, and we can apply induction. Finally, if C has length 6, then $w_i \neq w_{i+3}$ for at least one choice of $i = 0, 1, 2$, because otherwise C would go around a triangle twice, and so it would be (even, even). □

Lemma 3. *Let G be balanced, $uv \in E(G)$ be a red edge, and $G' = G - \{u, v\}$. For* x = 1, 2r *or* 2b, *assume that $G' \sim$ (cx), but G does not have coloring (cx). Then G contains a 4-circuit C with $uv \in E(C)$ which is either $(3, 1)$ or $(1, 3)$. Moreover, if* x = 2r, *then C can be chosen to be $(3, 1)$.*

Proof. By Lemma 2, G' does not and G does have a 4-circuit C that is (odd, odd), and that is $(3, 1)$ if x = 2r. Thus this C intersects $\{u, v\}$. Let the nodes of C be w_0, w_1, w_2, w_3. If C already contains the edge uv, then we are done. Otherwise $|\{u, v\} \cap V(C)|$ is either 1 or 2. Accordingly, we may distinguish the following cases.

- $u = w_0$, $v \notin V(C)$. Note that vw_2 is an edge. Now either uvw_2w_1 or uvw_2w_3 is (odd, odd), moreover if x = 2r, then, since C was $(3, 1)$, this new 4-circuit is also $(3, 1)$.

- $u = w_0$, $v = w_2$. Now $G[w_0, w_1, w_2, w_3] \simeq K_4$, and so it decomposes into three disjoint edge-pairs. Any two of these three edge-pairs give a 4-circuit. C itself gives rise to two of these edge-pairs, one of them is $(1, 1)$, and the other one is either $(2, 0)$ or $(0, 2)$, and it is $(2, 0)$ if x = 2r. The third edge pair, containing uv, is either $(2, 0)$ or $(1, 1)$. Clearly, it is possible to construct a 4-circuit containing uv, with the desired property on the number of colors.

□

Theorem 4 (Karzanov [3])**.** *Let $G \simeq K_{2n}$ or $K_{n,n}$. Then G has a perfect matching with k_R red and $k_B = n - k_R$ blue edges if and only if all of the following conditions hold.*

(t0) $k_R \leq \nu_R$ *and* $k_B \leq \nu_B$.

(t1) $G \sim$ (c1) $\Longrightarrow k_B \equiv \nu_B \pmod 2$ *(and then also* $k_R \equiv \nu_R \pmod 2$*).*

(t2r) $G \sim$ (c2r) $\Longrightarrow k_B \neq 1$.

(t2b) $G \sim$ (c2b) $\Longrightarrow k_R \neq 1$.

Proof. First note that for any graph G, the failure of any of (t0)–(t2) objects G having a (k_R, k_B)-matching. For the other direction, assume that G is a minimal balanced graph satisfying (t0)–(t2) without a (k_R, k_B)-matching. The reader is welcome to check that no such graph exists for $n \le 3$, so we have $n \ge 4$. By symmetry we may assume that $k_R \ge k_B$. Note that $0 < k_R < \nu_R$ since otherwise G clearly has a (k_R, k_B)-matching.

We try to choose an edge $uv \in E_R$ such that (t0) holds for $G' = G - \{u, v\}$, $k'_R = k_R - 1$, $k'_B = k_B$. Observe that $\nu'_R \ge \nu_R - 2$ so only $k'_B \le \nu'_B$ can fail. On one hand, if $\nu_B = n$, then $\nu'_B \ge n - 2 \ge \lfloor n/2 \rfloor \ge k_B = k'_B$. On the other hand, if $\nu_B < n$ then it is possible to choose $uv \in E_R$ such that $\nu'_B = \nu_B \ge k_B = k'_B$. So (t0) holds for G', k'_R, k'_B.

If G' had a (k'_R, k'_B)-matching, then together with the edge uv we would obtain a (k_R, k_B)-matching of G. So G' has no (k'_R, k'_B)-matching. Hence, by induction, G' violates (tx) for x = 1, 2r or 2b (recall that (t0) holds for G'), in particular, $G' \sim$ (cx). We prove that G itself does not have coloring (cx). Assume $G \sim$ (c1). Observe that $\nu'_B = \nu_B$ except if $\nu_B = n$, in which case it is easy to see from Figures 1 and 2 that $\nu'_B = \nu_B - 2$. Thus, since $k'_B = k_B$, (t1) would fail for G. The case x = 2r is trivial, because $k'_B = k_B$. Finally, in the case x = 2b we have $k_R = 2$; and if G had coloring (c2b), then it were easy to find a $(2, n-2)$-matching of G. Hence indeed, G itself does not have coloring (cx).

Therefore, we can apply Lemma 3, yielding a 4-circuit C of G with $uv \in E(C)$ which is (odd, odd), and which is $(3, 1)$ if $G' \sim$ (c2r). Note that C has a $(1, 1)$-matching. If $G - V(C)$ has a $(k_R - 1, k_B - 1)$-matching, then putting these together we obtain a (k_R, k_B)-matching of G and we are done. By Lemma 1, applied to G', k'_R, k'_B and the unique edge f of C spanned by G', the only case when $G - V(C)$ fails to have a $(k_R - 1, k_B - 1)$-matching is when $k_B = 1$. We finish the proof depending on x.

1. By Lemma 1, G' has an $(n-1, 0)$-matching, that is a perfect red matching. So, by Figures 1 and 2, the coloring of G' also satisfies (c2r). So the next point applies.

2r. Now C can be chosen to be $(3, 1)$. Moreover, by Lemma 1, $G - V(C)$ has a $(k_R - 2, k_B)$-matching. These together form the required (k_R, k_B)-matching of G.

2b. $k_R = k'_R + 1 = 2$ and so $4 \le n = k_R + k_B = 3$, which is impossible.

□

References

[1] F. Barahona and W.R. Pulleyblank. Exact arborescences, matchings and cycles. *Discrete Applied Mathematics*, 16(2):91–99, 1987.

[2] A. Galluccio and M. Loebl. On the theory of Pfaffian orientations. I. Perfect matchings and permanents. *Electronic Journal of Combinatorics*, 6(1):R6, 1999.

[3] A.V. Karzanov. Maximum matching of given weight in complete and complete bipartite graphs. *Cybernetics and Systems Analysis*, 23(1):8–13, 1987.

[4] L. Lovász. Randomized algorithms in combinatorial optimization. *Combinatorial optimization: papers from the DIMACS Special Year*, page 153, 1995.

[5] K. Mulmuley, U.V. Vazirani, and V.V. Vazirani. Matching is as easy as matrix multiplication. *Combinatorica*, 7(1):105–113, 1987.

[6] C.H. Papadimitriou and M. Yannakakis. The complexity of restricted spanning tree problems. *Journal of the ACM (JACM)*, 29(2):285–309, 1982.

[7] T. Yi, K.G. Murty, and C. Spera. Matchings in colored bipartite networks. *Discrete Applied Mathematics*, 121(1-3):261–277, 2002.