Two remarks on local edge-connectivity of graphs

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Abstract

We provide slight generalizations of a result of Lovász and a result of Hamidoune and Las Vergnas on local edge-connectivities.

1 Local edge-connectivity in directed graphs

Let $D = (V, A)$ be a directed graph. Multiple edges are allowed, but loops are not. For a set $X \subseteq V$, let $\boldsymbol{\delta(X)} = |\{xy \in A : x \in X,\ y \in V - X\}|$, $\boldsymbol{\varrho(X)} = \delta(V - X)$. For $u, v \in V$, the **local edge-connectivity** from u to v, $\boldsymbol{\lambda_D(u, v)}$, is defined as the maximum number of edge disjoint paths from u to v. By Menger's theorem, $\lambda_D(u, v) = \min\{\varrho(Y) : v \in Y \subseteq V - u\}$. A set Y providing the minimum is called $\boldsymbol{uv}$**-tight**. Note that for every vertex $v \neq u$, there exists a uv-tight set, and by the submodularity of ϱ, there exists a maximal such set U_v. We say that the vertex v is a **core** of U_v.

Proposition 1. *Let r be a vertex of a directed graph D such that $\delta(r) > \varrho(r)$. Then there exists a vertex s such that $\delta(s) < \varrho(s)$.*

Proof. Since $\sum_{v \in V}(\delta(v) - \varrho(v)) = 0$ and $\delta(r) - \varrho(r) > 0$, there exists a vertex s such that $\delta(s) - \varrho(s) < 0$. □

Theorem 1 (Lovász [2])**.** *Let r be a vertex of a directed graph D such that $\delta(r) > \varrho(r)$. Then there exists a vertex s such that $\lambda(r, s) > \lambda(s, r)$.*

The proof of Lovász [2] for Theorem 1 with a convenient modification provides the following common generalization of the above two results.

Theorem 2. *Let r be a vertex of a directed graph D such that $\delta(r) > \varrho(r)$. Then there exists a vertex s such that $\lambda(r, s) > \lambda(s, r)$ and $\delta(s) < \varrho(s)$.*

Proof. Let $T := \{v : \delta(v) < \varrho(v)\}$. Note that, by Proposition 1, T is not empty. Let $\mathcal{U} := \{U_1, \dots, U_k\} := \{U_v : v \in T\}$ be the set of maximal rv-tight sets where $v \in T$ and let $V_i := U_i \setminus \bigcup_{j \neq i} U_j$ for $i = 1, \dots, k$.

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Claim 1. *The core u_i of U_i belongs to V_i for all $i = 1, \dots, k$.*

Proof. If u_i belonged to U_j for some $j \neq i$, then by tightness, the submodularity of ϱ, $u_i \in U_i \cap U_j, u_j \in U_i \cup U_j$ and the maximality of U_i and U_j, we would have $\lambda(r,u_i) + \lambda(r,u_j) = \varrho(U_i) + \varrho(U_j) \geq \varrho(U_i \cap U_j) + \varrho(U_i \cup U_j) > \lambda(r,u_i) + \lambda(r,u_j)$, a contradiction. □

Let $V_0 := U_0 := V \setminus \bigcup_1^k U_i$.

Claim 2. $\delta(U_0) > \varrho(V_0)$.

Proof. By the definition of $\mathcal{U}$, we have $T \subseteq \bigcup_1^k U_i$, so for every $v \in U_0$, $\delta(v) - \varrho(v) \geq 0$. Moreover, $r \in U_0$ and $\delta(r) - \varrho(r) > 0$, thus $\delta(U_0) - \varrho(V_0) = \sum_{v \in U_0}(\delta(v) - \varrho(v)) > 0$. □

Suppose that $(*)$ $\lambda(r,v) \leq \lambda(v,r)$ for all $v \in T$.

Claim 3. $\sum_1^k \delta(V_i) \geq \sum_1^k \varrho(U_i)$.

Proof. For every $i = 1, \dots, k$, by $u_i \in V_i \subseteq V - r$, $(*)$ and the ru_i-tightness of U_i, we have $\delta(V_i) \geq \lambda(u_i, r) \geq \lambda(r, u_i) = \varrho(U_i)$, and the claim follows. □

Claim 4. $\sum_0^k \delta(V_i) \leq \sum_0^k \varrho(U_i)$.

Proof. Since $\{V_0, V_1, \dots, V_k\}$ is a subpartition of V, every edge is counted at most once on the left hand side. Let uv be an edge that contributes to the left hand side. Then there exists an index i such that $u \in V_i$ and $v \in V \setminus V_i = \bigcup_{j \neq i} U_j$ and hence there exists an index j such that $v \in U_j$. Since $u \notin U_j$, the edge uv contributes to the right hand side and the claim follows. □

Claims 2, 3 and 4 provide a contradiction. It follows that there exists a vertex $s \in T$ such that $\lambda(r,s) > \lambda(s,r)$ and the theorem is proved. □

2 Local edge-connectivity in regular bipartite graphs

Let $G = (V, E)$ be an undirected graph. Multiple edges are allowed, but loops are not. For a set $X \subseteq V$, let $\boldsymbol{d(X)}$= $|\{xy \in E : x \in X,\ y \in V - X\}|$. For $u, v \in V$, the **local edge-connectivity** from u to v, $\boldsymbol{\lambda_G(u,v)}$, is defined as the maximum number of edge disjoint paths from u to v. By Menger's theorem, $\lambda_G(u,v) = \min\{d(Y) : v \in Y \subseteq V - u\}$. A set Y providing the minimum is called $\boldsymbol{uv}$**-tight**. Note that for every vertex $v \neq u$, there exists a uv-tight set, and by the submodularity of d, there exists a minimal such set X_v. We say that the vertex v is a **core** of X_v. The graph G is called k**-regular** if each vertex of G is of degree k.

Proposition 2. *Let $G = (A, B; E)$ be a k-regular bipartite graph with $k \geq 1$. Then there exists a bijection $\{a_i b_i : a_i \in A, b_i \in B\}$ between A and B.*

Proof. $k|A| = \sum_{a \in A} d(a) = |E| = \sum_{b \in B} d(b) = k|B|$ and the proposition follows. □

Theorem 3 (Hamidoune and Las Vergnas [1]). *Let $G = (A, B; E)$ be a k-regular bipartite graph with $k \geq 1$. Then for every $a \in A$ there exists $b \in B$ such that $\lambda(a, b) = k$.*

The following theorem provides a common generalization of the above two results.

Theorem 4. *Let $G = (A, B; E)$ be a k-regular bipartite graph with $k \geq 1$. Then there exists a bijection $\{a_i b_i : a_i \in A, b_i \in B\}$ between A and B such that $\lambda(a_i, b_i) = k$ for all i.*

Let $H = (A, B; F)$ be the bipartite graph where for $a \in A$ and $b \in B$, $ab \in F$ if and only if $\lambda_G(a, b) \geq k$. It is well-known that the connected components $H_1, \dots, H_l$ are complete bipartite graphs. Theorem 4 is equivalent to the following Lemma. Let $A_i := V(H_i) \cap A$ and $B_i := V(H_i) \cap B$.

Lemma 1. *For every i, $|A_i| = |B_i|$.*

Proof. In the whole proof we consider the graph G. Note that by Proposition 2, $|A| = |B|$. Let a be an arbitrary vertex in A_i. Let $\mathcal{X}$ be the family $\{X_b : v \in B - B_i\}$ of the minimal ab-tight sets where $b \in B - B_i$. Note that $d(X) < k$ for every $X \in \mathcal{X}$.

Claim 5. *$\mathcal{X}$ is a laminar family.*

Proof. Suppose not, and let X_{b_1} and X_{b_2} be two crossing sets in $\mathcal{X}$.

First suppose that $X_{b_1} \cap X_{b_2}$ contains b_1 or b_2, say b_1. Then, by tightness, the submodularity of d, $b_1 \in X_{b_1} \cap X_{b_2}, b_2 \in X_{b_1} \cup X_{b_2}$ and the minimality of X_{b_1}, we have $\lambda(a, b_1) + \lambda(a, b_2) = d(X_{b_1}) + d(X_{b_2}) \geq d(X_{b_1} \cap X_{b_2}) + d(X_{b_1} \cup X_{b_2}) > \lambda(a, b_1) + \lambda(a, b_2)$, a contradiction.

Otherwise, $b_1 \in X_{b_1} \setminus X_{b_2}$ and $b_2 \in X_{b_2} \setminus X_{b_1}$. Then, by tightness, the submodularity of d, and the minimality of X_{b_1}, we have $\lambda(a, b_1) + \lambda(a, b_2) = d(X_{b_1}) + d(X_{b_2}) \geq d(X_{b_1} \setminus X_{b_2}) + d(X_{b_2} \setminus X_{b_1}) > \lambda(a, b_1) + \lambda(a, b_2)$, a contradiction. □

Claim 6. *For every $X \in \mathcal{X}$, $|X \cap A| = |X \cap B|$.*

Proof. Let $\alpha := d(X \cap A, B \setminus X)$ and $\beta := d(X \cap B, A \setminus X)$. Then, by $X \in \mathcal{X}$, $\alpha + \beta = d(X) < k$. Since $k|X \cap A| - \alpha = d(X \cap A, X \cap B) = k|X \cap B| - \beta$, we get that $|X \cap A| = |X \cap B|$. □

Let $\mathcal{X}^*$ be the maximal sets of $\mathcal{X}$. By Claim 5, the elements of $\mathcal{X}^*$ are disjoint. Let $X_A := \bigcup\{X \cap A : X \in \mathcal{X}^*\}$ and $X_B := \bigcup\{X \cap B : X \in \mathcal{X}^*\}$. By Claim 6, $|X_A| = \sum_{X \in \mathcal{X}^*} |X \cap A| = \sum_{X \in \mathcal{X}^*} |X \cap B| = |X_B|$. Note that $B_i = B - X_B$ and $A_i \subseteq A - X_A$ and hence, by $|A| = |B|$, we have $|A_i| \leq |B_i|$. Then $|A| = \sum_{i=1}^{l} |A_i| \leq \sum_{i=1}^{l} |B_i| = |B| = |A|$, and the lemma follows. □

References

[1] Y. P. Hamidoune, M. Las Vergnas, Local edge-connectivity in regular bipartite graphs, *J. Combin. Theory/Series B* **44** (1986) 370-371.

[2] L. Lovász, Connectivity in digraphs, *J. Combin. Theory/Series B* **15** (1973) 174-177.