On the list colouring of two matroids

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1 Introduction

Let M_1 and M_2 be two matroids on the same ground set S. Given a list of colours $L(s)$ for each $s \in S$, we say that M_1 and M_2 are *jointly list colourable* with lists $L(s)$ if one can choose a colour for each element from its list such that every monochromatic set is independent in both M_1 and M_2.

Definition 1. Let M_1 and M_2 be two matroids on the same ground set S, and let k denote the minimum number of common independent sets that cover S. We say that M_1 and M_2 have the *joint list colouring property* if they are jointly list colourable given any lists $L(s)$ with $|L(s)| \geq k$ $(s \in S)$.

We are not aware of any example of two matroids where the joint list colouring property does not hold, but very few cases are known. One famous example where the property holds is the bipartite edge list colouring theorem of Galvin [1], an equivalent form of which is that any two partition matroids have the joint list colouring property. Besides that, if the two matroids are the same, then they have the joint list colouring property, since it specializes to Seymour's result [3]. In this note we prove the property for some other simple cases.

2 Transversal matroids

Theorem 2. *Any two transversal matroids have the joint list colouring property.*

Proof. Let M_1 and M_2 be two transversal matroids on S, and suppose that S can be covered by k common independent sets. Let $G_1 = (S, T_1, E_1)$ and $G_2 = (S, T_2, E_2)$ be bipartite graphs whose transversal matroids are M_1 and M_2, respectively. Take disjoint common independent sets $I_1, I_2, \dots I_k$ covering S. Let P^i_j be a matching in E_i which covers I_j $(i = 1, 2,\ j \in [k])$. The bipartite graph $(S, T_i, \cup_{j\in[k]} P^i_j)$ is a subgraph of G_i, thus the independent sets in its transversal matroid M'_i are independent in M_i too $(i = 1, 2)$. Since the matroids M'_1 and M'_2 are partition matroids (with bounds equal to 1), it is enough to prove the theorem for such matroids.

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If M_1 and M_2 are partition matroids on S with bounds equal to 1, then there is a bipartite graph G whose edges correspond to S, and a subset of S is a common independent set if and only if the corresponding edges form a (partial) matching. Thus the list colourability follows from Galvin's theorem [1]. □

3 Two spanning arborescences

Theorem 3. *Let $D = (V, A)$ be a directed graph which is the disjoint union of two spanning r-arborescences ($r \in V$), and let the two matroids on ground set A be the following: M_1 the graphic matroid of D (obliviously to the orientation of the arcs), and M_2 the partition matroid with respect to the partition formed by the in-stars, with upper bounds equal to 1. Then the two matroids have the joint list colouring property.*

Proof. The condition that D is the union of two r-arborescences means that A is the disjoint union of two common bases of M_1 and M_2. So we have to show that if we are given lists $L(.)$ of size 2, then there is a proper list colouring. We have to find a colouring, in which there is no monochromatic directed cycle, and no two arcs of the same colour with the same head (this implies that there is no monochromatic cycle at all). We proceed by induction.

Due to the constructive characterization of rooted k-arc-connected digraphs (see [4, 2]), D can be obtained from a single node by repeating the following steps:

(a) adding a new node and two incoming arcs from existing nodes,

(b) dividing an arc with a new node and add an incoming arc.

If the last step of a construction of D was of type (a) from digraph D', then a proper list colouring can be obtained from one of D' (with the respective lists) by colouring the two new edges with different colours.

If the last step was of type (b) from digraph D', then let us denote the new node by z, the arc which was divided by $a = uv$, and the new arc by $b = wz$. Let the lists in D' be the same as in D on the common arcs, and $L(zv)$ on the arc a. From induction, there is a proper list colouring. Assume that a is blue. Then in D, colour each arc with the same colour, if it existed in D'', and colour the arc zv blue. Now colour the arc wz with a colour on its list different from blue, say green, and then colour the arc uz with a colour on its list different from green. This way no monochromatic directed cycle can arise, and the in-stars are also multicoloured. □

4 Matroids of rank 2

Theorem 4. *If both matroids M_1 and M_2 are of rank 2, then they have the joint list colouring property.*

Proof. Let us take a partition $\mathcal{P}$ of the ground set S consisting of k common independent sets (where k is the minimum size of such a partition), and take colour lists $L(.)$

of size k. If there is a singleton in $\mathcal{P}$, or there is a set of size 2 in $\mathcal{P}$ with a common colour on the lists of the two elements, then we can delete both the set and the colour from the other lists, and we are done by induction. So suppose that the sets in $\mathcal{P}$ are of size 2, and all pairs of colour lists are disjoint.

We claim that in this case the elements can be coloured with different colours. Using Hall's theorem, it is enough to prove that for any set $X \subset S$, $|\cup_{s\in X} L(s)| \geq |X|$. This is clear if $|X| \leq k$. If $|X| > k$, then X contains a set of the partition $\mathcal{P}$, thus $|\cup_{s\in X} L(s)| \geq 2k \geq |X|$. □

References

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