Computing the minimum cut in hypergraphic matroids

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Abstract

Hypergraphic matroids were defined by Lorea as generalizations of graphic matroids. We show that the minimum cut (co-girth) of a multiple of a hyper-graphic matroid can be computed in polynomial time.

It is well-known that the size of the minimum cut (co-girth) of a graph can be computed in polynomial time. For connected graphs, this is equivalent to computing the co-girth of the circuit matroid. On the other hand, it is NP-hard to determine the girth of a transversal matroid (see McCormick [6]), so the problem of finding the co-girth is hard for fairly simple matroid classes such as gammoids. It would be useful to have new classes of matroids where the problem is tractable. In the first section of this note we briefly review the known results on the co-girth of multiples of graphic matroids, and in the second section we show that the problem remains tractable for hypergraphic matroids.

1 Graphic matroids

A graph G = (V, E) is called *k*-partition-connected if

 $e_G(\mathcal{P}) \geq k(|\mathcal{P}| - 1)$ for every partition \mathcal{P} ,

where $e_G(\mathcal{P})$ denotes the number of cross-edges, i.e. edges intersecting at least 2 members of the partition \mathcal{P} . The graph G is (k, l)-partition-connected if it remains k-partition-connected after the deletion of any l edges, or in other words,

 $e_G(\mathcal{P}) \ge k(|\mathcal{P}| - 1) + l$ for every non-trivial partition \mathcal{P} .

If M is the circuit matroid of the graph G = (V, E), we call kM (the matroid sum of k copies of M) the k-circuit matroid of G. The rank of kM is

$$r_{kM}(E) = \min\{k(|V| - |\mathcal{P}|) + e_G(\mathcal{P}): \mathcal{P} \text{ is a partition of } V\}.$$
 (1)

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This means that the rank is k(|V|-1) if and only if G is k-partition-connected. In this case, the co-girth of kM is at least l+1 if and only if G is (k, l)-partition-connected. On the other hand, if G is connected but not k-partition-connected, then the co-girth is 1, since every cross-edge of the partition giving the minimum in (1) is a co-loop.

The orientation theorems of Frank [1] imply that (k, l)-partition-connectivity can be decided in polynomial time. One consequence is that the co-girth of a k-circuit matroid can be computed in polynomial time: we can find the smallest l for which Gis (k, l)-partition-connected by binary search.

2 Hypergraphic matroids

Hypergraphic matroids were introduced by Lorea [5] as a generalization of graphic matroids. Given a hypergraph H = (V, E), a subset $F \subseteq E$ of hyperedges is a hyperforest if one can choose two nodes from each hyperedge of F so that the resulting graph is a forest. The *circuit matroid* of H is the matroid on ground set E whose independent sets are the hyperforests.

If M is the circuit matroid of H = (V, E), then kM is called the k-circuit matroid of H. The following result appeared in [3].

Theorem 1 (Frank, Király, Kriesell [3]). The rank of the k-circuit matroid of a hypergraph H = (V, E) is

$$\min\{k(|V| - |\mathcal{P}|) + e_H(\mathcal{P}): \mathcal{P} \text{ is a partition of } V\},$$
(2)

where $e_H(\mathcal{P})$ denotes the number of hyperedges intersecting at least 2 members of the partition \mathcal{P} .

Analogously to graphs, we say that the hypergraph H is *k*-partition-connected if

 $e_H(\mathcal{P}) \geq k(|\mathcal{P}| - 1)$ for every partition \mathcal{P} ,

and (k, l)-partition-connected if

 $e_H(\mathcal{P}) \ge k(|\mathcal{P}| - 1) + l$ for every non-trivial partition \mathcal{P} .

It follows from Theorem 1 that the k-circuit matroid of H has rank k(|V| - 1) if and only if H is k-partition-connected. Moreover, if H is connected but not k-partitionconnected, then its co-girth is 1, since any cross-hyperedge of the partition giving the minimum in (2) is a co-loop.

In the following we assume that H is k-partition-connected, so the co-girth of the k-circuit-matroid is at least l + 1 if and only if H is (k, l)-partition-connected. To determine the co-girth, we have to find the smallest l for which H is (k, l)-partition-connected. This can be done by binary search if we can decide for a given l whether H is (k, l)-partition-connected.

Although the main tools for solving this problem have been described in [2], the paper did not explicitly address algorithmic aspects. The purpose of this note is to

show that there is a polynomial algorithm to decide (k, l)-partition-connectivity of a hypergraph. We do not aim to present a particularly fast algorithm; it remains an interesting question how hard this problem is compared to finding the minimum cut of a graph.

Theorem 2. Let H be a k-partition-connected hypergraph, and l a positive integer. It can be decided in polynomial time whether H is (k, l)-partition-connected.

Proof. The proof uses the notion of directed hypergaph. For definitions of hypergraph orientations and (k, l)-arc-connectivity of directed hypergraphs, see [2].

Case 1: $k \ge l$. The solution of this case is implicitly described in [2]. It is shown there that H is (k, l)-partition-connected if and only if it has a (k, l)-arc-connected orientation. Moreover, the paper shows that finding a (k, l)-arc-connected orientation of a hypergraph amounts to finding an integer vector in a base polyhedron defined by a crossing supermodular function (or proving that the polyhedron is empty), which can be done in polynomial time.

Case 2: k < l. In this case the characterization of (k, l)-partition-connectivity is more complicated. It is described by the following result of [2].

Lemma 3 (Frank, Király, Király [2]). Let $k \leq l$ be positive integers. A hypergraph H = (V, E) is (k, l)-partition-connected if and only if for every pair $s, t \in V$ it has a k-arc-connected orientation where there are l arc-disjoint paths from s to t. \Box

By the lemma, it suffices to check whether for a given node pair $s, t \in V$ there is a k-arc-connected orientation of H where there are l arc-disjoint paths from s to t. We show that this problem can be solved by finding an integer vector in the intersection of 2 base polyhedra.

Let us define the following two set functions:

$$p_1(X) = \begin{cases} 0 & \text{if } X = \emptyset \text{ or } X = V, \\ -\infty & \text{if } s \in X \subseteq V - t, \\ l & \text{if } t \in X \subseteq V - s, \\ k & \text{otherwise,} \end{cases}$$
$$p_2(X) = \begin{cases} 0 & \text{if } X = \emptyset \text{ or } X = V, \\ -\infty & \text{if } t \in X \subseteq V - s, \\ k & \text{otherwise.} \end{cases}$$

It is easy to see that both p_1 and p_2 are intersecting supermodular, so the set functions defined by $q_1(X) = p_1(X) + i_H(X)$ and $q_1(X) = p_2(X) + i_H(X)$ are also intersecting supermodular (where $i_H(X)$ denotes the number hyperedges of H induced by X). Consequently, the following two polyhedra are base polyhedra:

$$B(q_1) = \{ x \in \mathbb{R}^V : \sum_{v \in V} x_v = |E|, \sum_{v \in X} x(v) \ge q_1(X) \text{ for every } X \subseteq V \},\$$

$$B(q_2) = \{ x \in \mathbb{R}^V : \sum_{v \in V} x_v = |E|, \sum_{v \in X} x(v) \ge q_2(X) \text{ for every } X \subseteq V \}.$$

An orientation D of H satisfies our requirements (it is k-arc-connected and has l arc-disjoint paths from s to t) if and only if $\rho_D(X) \ge \max\{p_1(X), p_2(X)\}$ for every $X \subseteq V$, or equivalently,

$$\sum_{v \in X} \varrho_D(v) \ge \max\{q_1(X), q_2(X)\} \text{ for every } X \subseteq V.$$

To finish the proof, we use the following easy lemma from [2].

Lemma 4. Given a hypergraph H = (V, E) and a vector $x \in \mathbb{Z}_+^V$, there is an orientation D of H for which $\varrho_D(v) = x_v$ for every $v \in V$ if and only if $\sum_{v \in V} x_v = |E|$ and $\sum_{v \in X} x(v) \ge i_H(X)$ for every $X \subseteq V$.

The lemma implies that there is an orientation D of H that satisfies our requirements if and only if there is a vector $x \in \mathbb{Z}_+^V$ such that $\sum_{v \in V} x_v = |E|$ and $\sum_{v \in X} x(v) \ge \max\{q_1(X), q_2(X)\}$ for every $X \subseteq V$, since $\max\{q_1(X), q_2(X)\} \ge i_H(X)$ for every $X \subseteq V$. Therefore, in order to find the orientation, we have to find an integer vector in $B(q_1) \cap B(q_2)$, or deduce that no such vector exists. This can be done in polynomial time since $B(q_1)$ and $B(q_2)$ are base polyhedra given by intersecting supermodular functions, and a polynomial time evaluation oracle as required in [4] is available.

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