Node-to-area connectivity augmentation of hypergraphs without increasing the rank

Attila Bernáth*

Abstract

The *rank-respecting node-to-area connectivity augmentation problem of hypergraphs* is the following: given a hypergraph $H = (V, \mathcal{E})$ of rank at most γ, a collection of subsets $\mathcal{W}$ of V and a requirement function $r : \mathcal{W} \to \mathbb{Z}_+$, find a hypergraph H' of minimum total size such that $\lambda_{H+H'}(x, W) \geq r(W)$ for any $W \in \mathcal{W}$ and $x \in V$ and the rank of H' is at most γ. This problem was investigated by Ishii and Hagiwara for $\gamma = 2$ (i.e. graphs). Though the problem is NP-complete (even if H is the empty graph and $r(W) = 1$ for every $W \in \mathcal{W}$), they observed that the assumption $r \geq 2$ surprisingly makes the problem tractable and they gave a polynomial algorithm solving it in the $\gamma = 2$ case. In this note we solve this problem for any $\gamma \geq 3$.

1 Introduction

Let us define the following *rank-respecting node-to-area connectivity augmentation problem of hypergraphs*: given a hypergraph $H = (V, \mathcal{E})$ of rank at most γ, a collection of subsets $\mathcal{W}$ of V and a function $r : \mathcal{W} \to \mathbb{Z}_+$, find a hypergraph H' of minimum total size such that $\lambda_{H+H'}(x, W) \geq r(W)$ for any $W \in \mathcal{W}$ and $x \in V$ and the rank of H' is at most γ. This problem was investigated by Ishii and Hagiwara in [2] for $\gamma = 2$ (i.e. graphs). Though the problem is NP-complete (even if H is the empty graph and $r(W) = 1$ for every $W \in \mathcal{W}$), they observed that the assumption $r \geq 2$ surprisingly makes the problem tractable and they gave a polynomial algorithm solving it. The problem for an arbitrary γ was introduced in [1], where we have shown that a greedy algorithm almost gives an optimal solution for the problem: the algorithm shown there can only fail by at most one for this problem, in other words, the total size of the solution provided can be at most one more than the optimum, if $\gamma \geq 3$. In this note I want to show that a slight modification of that algorithm gives an optimal solution to the problem.

Throughout I will use the notations introduced in [1]. This note does not want to be self-contained: please read Sections 1,2,3 and 5 of [1] to become familiar with the preliminaries.

*MTA-ELTE Egerváry Research Group, Department of Operations Research, Eötvös University, Pázmány Péter sétány 1/C, Budapest, Hungary, H-1117. E-mail: bernath@cs.elte.hu. Supported by OTKA grants K60802.

We will need the following observation about a crossing negamodular set function q (where the notation $M_q = \max\{q(X) : X \subseteq V\}$ is used for any set function q).

$$\text{If } X, Y \subseteq V \text{ are crossing, } q(X) = \overline{q}(Y) = M_q, \text{ then } q(X \cap Y) = 1 \text{ and } \overline{q}(X \cup Y) = 1. \tag{1}$$

The following claim generalizes this statement in one direction. It is proved by a simple induction.

Claim 1. *Let $q : 2^V \to \mathbb{Z}$ be crossing negamodular and assume that $X_0, X_1, X_2, \ldots, X_t$ are subsets of V (where $t \geq 0$) such that $\overline{q}(X_0) = q(X_1) = q(X_2) = \cdots = q(X_t) = M_q$ and X_i crosses $X_0 \cup \bigcup_{j<i} X_j$ for any $i = 1, 2, \ldots, t$. Then $\overline{q}(X_0 \cup \bigcup_{j \leq t} X_j) = M_q$.*

2 The result

The generalization of the above problem that I will consider is the following: assume that we are given a crossing negamodular set function $R : 2^V \to \mathbb{Z}$ that does not take 1 as value. We will assume that R is given through a function-evaluation oracle. Let furthermore $H = (V, \mathcal{E})$ be an arbitrary hypergraph of rank at most γ. Consider the problem of finding a hypergraph H' that covers $q = R - d_H$, it has rank at most γ and it has smallest possible total size. Let us call this problem the *main problem.* In this note I want to solve this problem for any $\gamma \geq 3$. The Algorithm GREEDYCOVER given in [1] outputs a hypergraph covering q that has minimum total size, however its rank may be bigger than γ (when it is $\gamma + 1$). We suggest the following simple modification of that algorithm to solve our main problem. Assume that the Algorithm GREEDYCOVER (with input $p = (R - d_H)^s$ and a minimal $m \in C(p) \cap \mathbb{Z}^V$) did not output a feasible hypergraph for this problem. Let the output of the algorithm be $G + e$ (where G is a graph, e is a hyperedge of size $\gamma + 1$). Our idea is the following: if the graph G does not contain edges at all, then it is easy to see that the greedy bound cannot be achieved, but one more is already enough (and it is simple to see how to reach it). Otherwise, if G contains an edge ab, then an appropriate node $c \in e$ can be deleted from e and added to ab (thus creating a hyperedge $\{a, b, c\}$ of size 3) and the hypergraph $H' = (V, E(G) - \{ab\} + \{a, b, c\} + e')$ of total size meeting the greedy bound is a feasible solution, where $e' = e - c$. In what follows we show that almost any choice of c will be good. In the rest of this note we assume that we are at the stuck situation of the Algorithm GREEDYCOVER (so the notations p, m, V^+ are meant for this case). We will denote the output of the algorithm by $G + e$ (where G is a graph and e is a hyperedge) and V^+ will just be a synonym for e. The following lemma tells us the condition that we have to satisfy when choosing node c.

Lemma 1. *Let $p = p_0 - d_G$ with a symmetric positively skew-supermodular function p_0 and let $m \in C(p)$. Assume that there does not exist an admissible splitting-off. Let $ab \in E(G)$ and $c \in V^+$ be arbitrary. Let the hypergraph G' be obtained from G by deleting the edge ab and adding the 3-hyperedge $\{a, b, c\}$, let $p' = p_0 - d_{G'}$ and let $m' = m - \chi_{\{c\}}$. Then $m' \in C(p')$ if and only if there is no set $X \subseteq V$ satisfying $p(X) = m(X) = 1$, $c \in X$ and $\{a, b\} \cap X \neq \emptyset$.*

Proof. If $m' \notin C(p')$ then there must be a set $X \subseteq V$ such that $m'(X) < p'(X)$. Since there was no admissible splitting-off, $p \leq 1$ and it is easy to check that $p' \leq p$, implying that $p'(X) = p(X) = 1$. This together with $m \in C(p)$ gives that $m(X) \geq p(X) = p'(X) = 1 > m'(X)$, implying that $c \in X$ and $\{a, b\} \cap X \neq \emptyset$, as claimed. □

We want to characterize the situation when the Algorithm GREEDYCOVER fails solving our main problem. Recall that for a pair $u, v \in V^+$, the unique minimal set blocking them is denoted by X_{uv}.

Claim 2. *If there is no admissible splitting-off, tight sets are singletons and for four nodes $v_1, v_2, v_3, v_4 \in V^+$ the sets $X_{v_1v_2}, X_{v_2v_3}$ and $X_{v_3v_4}$ are all of the same type then $|X_{v_1v_2}| = |X_{v_2v_3}| = |X_{v_3v_4}| = 2$.*

The following lemma was shown in [1], we recall the proof for sake of completeness. A *large hyperedge* is a hyperedge that contains at least 2 positive nodes.

Lemma 2. *If there is no admissible splitting-off, tight sets are singletons and $m(V) \geq 5$ then there exists a large hyperedge. Furthermore, the number of positive nodes that are avoided by a large hyperedge is at most one.*

Proof. Assume that there is no large hyperedge. By the minimality of m, an arbitrary $x \in V^+$ is contained in a non-singleton hyperedge e. We claim that neither the q-graph nor the $\overline{q}$-graph can contain a path consisting of 3 edges. Assume indirectly that for some four nodes $x, y, u, v \in V^+$ the sets X_{xy}, X_{yu}, X_{uv} are all of the same type: then Claim 2 gives that they all are of cardinality 2. But then X_{xy} and X_{yu} cannot satisfy $(-)$ with equality by the nonsingleton hyperedge containing y, proving our claim. One can check that the edge set of a complete graph on at least 5 nodes cannot be decomposed into 2 sets such that neither of them contains a path of 3 edges, so there must be a large hyperedge.

Assume that there is a large hyperedge f that avoids $x \in V^+$. Since f is large, there exist $u, v \in V^+ \cap f$. X_{xu} and X_{xv} must be of the same type by the crossing negamodularity. If f avoids another positive node y then X_{xu} and X_{yu} cannot be of the same type for similar reasons. This implies that f cannot avoid a third positive node, so it contains at least 3 positive nodes, since $m(V) \geq 5$. Then the type of X_{uv} and X_{ux} must be different, since they cannot satisfy $(-)$ with equality because of the edge f that is not contained in X_{uv}. But then the type of X_{uv} and X_{uy} would be the same, which cannot hold for the same reason, so f cannot avoid the second positive node y. Furthermore, these observations on the $q\overline{q}$-graph show that x can be the only positive node that is avoided by a large hyperedge. □

The following corollary can be read out from the proof.

Corollary 3. *If there is no admissible splitting-off, tight sets are singletons and $m(V) > \gamma \geq 4$ then there exists a large hyperedge f and a node $x \in V^+$ such that $f = V^+ - x$. In this case either the q-graph or the $\overline{q}$-graph is the complete graph on $V^+ - x$ and the other graph is the complement (i.e. a star) centered at x.*

We mention that if $m(V) = 4$ then we don't necessarily have large hyperedges: an example can be found in [2]. One can also check that even if there are large hyperedges, they might contain 2 positive nodes if $m(V)$ is only 4. However, the second statement of Corollary 3 still holds.

Lemma 4. *If there is no admissible splitting-off, tight sets are singletons and $m(V) = 4$ and $\gamma \leq 3$ then there exists a special node $x \in V^+$ such that either the q-graph or the $\overline{q}$-graph is the complete graph on $V^+ - x$ and the other graph is the complement (i.e. a star) centered at x.*

Proof. We have to prove that neither the q-graph, nor the $\overline{q}$-graph contains a path of 3 edges. Assume that for four nodes $v_1, v_2, v_3, v_4 \in V^+$ the sets $X_{v_1v_2}, X_{v_2v_3}$ and $X_{v_3v_4}$ are all of the same type. By Claim 2 then $|X_{v_1v_2}| = |X_{v_2v_3}| = |X_{v_3v_4}| = 2$. Since $p(X_{v_2v_3}) = 1$, a hyperedge h of H leaves the set $X_{v_2v_3}$. Assume wlog. that h contains v_2: since $X_{v_2v_3}$ and $X_{v_1v_2}$ satisfies $(-)$ with equality, h must contain v_1, too, and h cannot contain v_4 (note that h cannot contain all the four nodes v_1, v_2, v_3, v_4). Since $X_{v_2v_3}$ and $X_{v_3v_4}$ satisfies $(-)$ with equality, h cannot contain v_3 (in fact we proved that $h = v_1v_2$). Because of the edge h, the type of $X_{v_2v_3}$ and $X_{v_1v_3}$ must be the same. Since $p(v_3) = 1$, there must be an edge g of H leaving v_3: this edge cannot leave $\{v_2, v_3, v_4\}$, since $(-)$ must hold with equality for $X_{v_2v_3}$ and $X_{v_3v_4}$. In any case, either $X_{v_2v_3}$ and $X_{v_1v_3}$, or $X_{v_1v_3}$ and $X_{v_3v_4}$ will not satisfy $(-)$ with equality. □

In the lemmas above we have shown that there exists a unique positive node x such that either the q-graph or the $\overline{q}$-graph is the complete graph on $V^+ - x$ and the other graph is the complement (i.e. a star). This translated to the situation before contraction means, that for any $u \in V^+$ there exists a tight set $X(u)$ that was contracted (so these sets are disjoint).

In the $\gamma \geq 4$ **case** $X(u) \cup X(v)$ is dangerous for any $u, v \in V^+ - x$. Furthermore there exists at least one large hyperedge $f \in H$ that has size γ and satisfies that $|f \cap X(u)| = 1$ for any $u \in V^+ - x$. Let $Y = V - \cup_{u \in V^+ - x} X(u)$, so $X(x) \subsetneq Y$ (it must be a proper subset, since $p(Y) \neq 1$, since no hyperedge leaves Y). In this case it is not hard to see that every set $Y \cup X(u)$ $(u \in V^+ - x)$ is of $\overline{q}$-type. It is clear, that the graph G does not have edges between the partition classes $\{X(u) : u \in V^+ - x\} \cup \{Y\}$.

On the other hand, **in the** $\gamma \leq 3$ **case** there exists a set $Z \subseteq V - \cup_{u \in V^+} X(u)$ such that $X(u) \cup X(v) \cup Z$ is dangerous for any $u, v \in V^+ - x$ and these sets all have the same type (Z is empty in the $\gamma > 3$ case). Let $Y = V - Z - \cup_{u \in V^+ - x} X(u)$: again $X(x) \subsetneq Y$ (it must be a proper subset, since $p(Y) \neq 1$, since no hyperedge leaves Y). In this case the edges of G are either induced in a member of the partition $\{X(u) : u \in V^+ - x\} \cup \{Y, Z\}$, or they can even go between two classes (but only between $X(u)$ and $X(v)$ or between $X(u)$ and Z, where $u, v \in V^+ - x$). In both cases we can prove the following lemma.

Lemma 5. *The sets $X(u)$ $(u \in V^+ - x)$ are maximal tight sets.*

Proof. Assume that there is $u \in V^+ - x$ and a tight set $X \supsetneq X(u)$. Let $v, w \in V^+ - \{u, x\}$ be arbitrary and observe that type of $X(v) \cup X(w) \cup Z$ and that of $Y \cup X(v) \cup X(w) \cup Z$ is different: this follows from (1) applied to $X(v) \cup X(w) \cup Z$

and $Y \cup X(v)$. This implies that $X \cap X(v) = X \cap Z = \emptyset$, since ($\cap\cup$) for X and either of $X(v) \cup X(w) \cup Z$ and $Y \cup X(v) \cup X(w) \cup Z$ (and p) would give a contradiction. We only need to prove that $X \cap Y$ is empty. Assume that it is not and distinguish the following two cases. Clearly, the type of X and $Y \cup X(z)$ has to be the same for any $z \in V^+ - x$, otherwise $p(X \cap (X(z) \cup Y)) = 1$ would follow from ($\cap\cup$) for X and $X(z) \cup Y$, and it would give a contradiction.
CASE I: Every set $Y \cup X(z)$ ($z \in V^+ - x$) is of q-type. By (1) applied to $X(v) \cup X(w) \cup Z$ and $Y \cup X(v)$, the set $Y \cup X(v) \cup X(w) \cup Z$ would be of $\overline{q}$-type, implying that $q(X \cap Y) = 1$, a contradiction.
CASE II: Every set $Y \cup X(z)$ ($z \in V^+ - x$) is of $\overline{q}$-type. Apply Claim 1 for $X_0 = X$ and the sets $X(u) \cup X(z) \cup Z$ ($z \in V^+ - x$) to get that $p(X \cup Z \cup \bigcup_{z \in V^+ - x} X(z)) = 1$. Let $Y' = V - (X \cup Z \cup \bigcup_{z \in V^+ - x} X(z)) = Y - X$: it has p-value 1, so there must be a hyperedge h leaving it. This hyperedge cannot leave Y, so it enters $Y \cap X$. But in this case the sets $X \cup X(v) \cup Z$ and $X \cup X(w) \cup Z$ could not satisfy ($-$) with equality, though both of them have $\overline{q}$-value 1. This contradiction finishes the proof of this lemma. □

This Lemma shows that our idea can be implemented the following way: if G contains an edge ab, and $c \in V^+ - x$ is such that $X(c) \cap \{a, b\} = \emptyset$ then the hypergraph $H' = (V, E(G) - \{ab\} + \{a, b, c\} + e')$ of total size meeting the greedy bound is a feasible solution, where $e' = V^+ - c$. Let us give the pseudocode of the modified version of the Algorithm GREEDYCOVER that we have suggested.

Algorithm NEGAMODULAR_COVER
begin
    INPUT A **crossing negamodular** function $R : 2^V \to \mathbb{Z}$ (given with an oracle) that satisfies $R(X) \neq 1$ for any $X \subseteq V$, and a hypergraph $H = (V, \mathcal{E})$ of rank at most γ (where $\gamma \geq 3$)
    OUTPUT A hypergraph $H' = (V, \mathcal{E}')$ covering $R - d_H$ having smallest total size and rank $\leq \gamma$
1.1. Let $q = R - d_H$ and $p = q^s$ and find a minimal $m \in C(p) \cap \mathbb{Z}^V$
1.2. Initialize $H' = (V, \emptyset)$
1.3. While there exists an admissible pair u, v do
1.4.   Let $m = m - \chi(u) - \chi(v)$ and $p = p - d_{(V,\{uv\})}$ and $H' = H' + uv$
1.5. EndWhile
1.6. If $m(V) \leq \gamma$ then let $H' = H' + e$ where $\chi_e = m$
1.7. Else (i.e. $m(V) = \gamma + 1$)
1.8.   If $E(H') = \emptyset$ then let $E(H') = \{ab, V^+ - b\}$ with arbitrary $a, b \in V^+$
1.9.   Else
1.10.   Let $ab \in E(H')$ be arbitrary and $c \in V^+ - x$ such that $X(c) \cap \{a, b\} = \emptyset$ (where $x \in V^+$ is the special node given by Corollary 3 and Lemma 4)
1.11.   Let $E(H') = E(H') - ab + \{\{a, b, c\}, \{V^+ - c\}\}$
1.12.   EndIf
1.13. EndIf
1.14. Output H' and STOP
end

References

[1] Attila Bernáth and Tamás Király, *A new approach to splitting-off*, Tech. Report TR-2008-02, Egerváry Research Group, Budapest, 2008, `www.cs.elte.hu/egres`.

[2] Toshimasa Ishii and Masayuki Hagiwara, *Minimum augmentation of local edge-connectivity between vertices and vertex subsets in undirected graphs*, Discrete Appl. Math. **154** (2006), no. 16, 2307–2329.