A conjecture on hypergraph orientation

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Abstract

We propose a conjecture on the orientation of hypergraphs which have the property that no hyperedge intersects minimally a regular family of sets. The truth of the conjecture would imply that non-perfect graphs are not kernel-solvable – the only known proof of which is based on the Strong Perfect Graph Theorem. We show that the conjecture is true if the hypergraph contains a connected graph.

Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph. For $Z \subseteq V$, we introduce the following notation:

$$f(Z) := \min\{|X \cap Z| : \ X \in \mathcal{E}\},$$
$$\mathcal{E}_Z := \{X \in \mathcal{E} : \ |X \cap Z| = f(Z)\}.$$

Let us propose the following conjecture.

Conjecture 1. *Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph. Suppose that $\cap_{Z \in \mathcal{F}} \mathcal{E}_Z = \emptyset$ for any regular hypergraph $(V, \mathcal{F})$ where $V \notin \mathcal{F}$. Then there is a function $h : \mathcal{E} \to V$ such that $Z - h(\mathcal{E}_Z) \neq \emptyset$ for every $\emptyset \neq Z \subsetneq V$, where $h(\mathcal{E}_Z)$ denotes*

$$\{v \in V : \ \exists X \in \mathcal{E}_Z, \ h(X) = v\}.$$

The function h must satisfy $h(X) \in X$ for every $X \in \mathcal{E}$, since $h(X) = v \notin X$ would mean that for $Z = \{v\}$ we have $X \in \mathcal{E}_Z$ and thus $Z \subseteq h(\mathcal{E}_Z)$. In this light, h can be interpreted as an assignment of a head to each hyperedge, so the conjecture concerns the orientation of hypergraphs.

A *kernel* in a directed graph $D = (V, E)$ is a stable set S with the property that for every node $u \in V - S$ there is an arc $uv \in E$ with $v \in S$. A *superorientation* of an undirected graph $G = (V, E)$ is a directed graph obtained from G by replacing each edge by an arc or two oppositely directed arcs. An undirected graph $G = (V, E)$ is *kernel-solvable* if every superorientation $(D = V, A)$ either has a kernel or there is a clique K such that $D[K]$ does not have a kernel. Berge and Duchet conjectured that kernel-solvable graphs are precisely the perfect graphs. The direction that perfect graphs are kernel-solvable has been proved by Boros and Gurvich [1]. The

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opposite direction was settled with the proof of the Strong Perfect Graph Theorem [2]. However, it is still interesting whether a proof that does not use the SPGT can be found.

Our conjecture would imply that non-perfect graphs are not kernel solvable without relying on the Strong Perfect Graph Theorem.

Proof. Let $G = (V, E)$ be a minimally imperfect graph, and let $H = (V, \mathcal{E})$ be the hypergraph whose hyperedges are the complements of the stable sets of G. If $(V, \mathcal{F})$ is a regular hypergraph not having V as a hyperedge, then $\alpha(G[Z]) = \overline{\chi}(G[Z])$ for every $Z \in \mathcal{F}$, but $\alpha(G) < \overline{\chi}(G)$, so for any stable set S there is a set $Z \in \mathcal{F}$ for which $S \cap Z$ is not a maximum stable set in Z. This means that the condition of the conjecture holds.

Let h be the function given by the conjecture. For a stable set S, let $v_S = h(V - S)$. Thus v_S is a node not in S, and the following property holds:

For any clique K, there is a node $u_K \in K$ such that $u_K \neq v_S$ if $|K \cap S| = 1$.

Let $D = (V, A)$ be the superorientation of G obtained by adding an arc $u_K v$ for every clique K and every node $v \in K$. Clearly, $D[K]$ has a kernel for every clique K. We claim that D does not have a kernel. Indeed, for every stable set S, there is no arc from v_S to S, so S cannot be a kernel. □

It is straightforward to see that the conjecture is true if $\mathcal{E}$ contains a singleton: in this case $\mathcal{E}_V = \{\{v\} : v \in V\}$, since otherwise $\mathcal{F} = \{u, V - u\}$ for some $\{u\} \notin \mathcal{E}$ would violate the condition. Thus $f(Z) = 0$ for every $Z \subsetneq V$, so any function h with the property $h(X) \in X$ $(X \in \mathcal{E})$ will work.

Theorem 2. *The conjecture is true if $\mathcal{E}$ contains a connected graph.*

Proof. In this case the hyperedge set $\mathcal{E}_V$ is a connected graph on the node set V. Let G denote this graph. First let us consider the case when G contains an odd cycle.

Case 1. G is non-bipartite.

Let C be an induced odd cycle in G. Our aim is to choose $h(X) \in X$ for every $X \in \mathcal{E}$, in such a way that the required property holds for every $\emptyset \neq Z \subsetneq V$.

Let us denote the nodes of C in cyclic order by $u_1, u_2, \ldots, u_k$. If $X = \{u_i, u_{i+1}\}$, then let $h(X) = u_{i+1}$ (cyclically). If $X \in \mathcal{E}$ contains a node in $V - V(C)$, then let $h(X)$ be such a node.

The rest of the hyperedges in $\mathcal{E}$ are subsets of $V(C)$ of size at least 3 (since C is an induced odd cycle of G). Let X be such a hyperedge. The nodes of X divide the cycle C into paths and an odd number of paths have odd length. We have two cases for the choice of $h(X)$:

1. If there are at least 3 paths of odd length, then let $h(X)$ be the node u_i in X which has the smallest index.
2. If there is exactly 1 path of odd length, then let $h(X)$ be the node u_i in X of minimum index that is not the starting node (according to the cyclic order) of the path of odd length.

Suppose that Z is a nonempty proper subset of V. If there is a hyperedge in $\mathcal{E}$ disjoint from Z, then $f(Z) = 0$, so the required property obviously holds for Z. Therefore we can assume that Z intersects all hyperedges in $\mathcal{E}$.

If $f(Z) = 2$, then Z would contain every element of V, since the graph G has no isolated nodes. Therefore we can assume that $f(Z) = 1$.

We have to prove that there is an element $z \in Z$ such that if $z = h(X)$ for some $X \in \mathcal{E}$, then $|Z \cap X| > 1$. Suppose for contradiction that this is not the case: for every element $z \in Z$ there is a hyperedge $X_z \in \mathcal{E}$ such that $z = h(X_z)$ and $|Z \cap X_Z| = 1$. Let $u_i \in Z$ be the node with the largest index such that X_{u_i} is not an edge of C. Clearly such a node u_i must exist: Z covers all edges of the odd cycle C, so it must contain two consecutive nodes of the cycle (say u_{j-1}, u_j), and X_{u_j} cannot be an edge of C, because the only possible edge is $u_{j-1}u_j$ which intersects Z in two nodes.

We obtained that X_{u_i} is a subset of $V(C)$ that has at least 3 elements. We distinguish two cases.

1. X_{u_i} defines at least 3 paths of odd length. Then u_i is the node of X_{u_i} with the smallest index. By the choice of u_i, X_{u_j} is an edge of C if $u_j \in Z$ and $j > i$. This is only possible if the nodes $u_i, u_{i+1}, \ldots, u_k$ are alternately in Z and not in Z. But X_{u_i} defines at least two odd paths on this part of the cycle, which means that at least one node u_j $(j > i)$ is in $X_{u_i} \cap Z$. Thus $|X_{u_i} \cap Z| \geq 2$, contradicting the choice of X_{u_i}.

2. X_{u_i} defines exactly one path of odd length. Then u_i may not be the node of X_{u_i} with the smallest index, but we know that there is at most one node with smaller index, so there is at least one node with larger index. Furthermore, we know that the path defined by u_i and the next node of X_{u_i} is an even path. By the same argument as in the previous case, the nodes u_j $(j \geq i)$ are alternately in Z and not in Z. So the evenness of the path means that the node following u_i in X_{u_i} is also in Z, hence $|X_{u_i} \cap Z| \geq 2$, contradicting the choice of X_{u_i}.

We proved that any Z that does not satisfy the property in the conjecture must contain all nodes in V, which completes the proof of the case when G is non-bipartite.

Case 2. G is bipartite.

Let the two colour classes of G be red and blue. The condition of the conjecture implies that at least one class, say red, contains a hyperedge in $\mathcal{E}$. For simplicity, we call such hyperedges in $\mathcal{E}$ *red hyperedges*.

For every red hyperedge X, let us consider a Steiner tree of minimum size for X in G. We choose one red hyperedge X_0 for which this Steiner tree is of minimum size. In addition, we choose a minimum size Steiner tree T_0 for X_0 that has the shortest possible path between two leaves. Let these two leaves be v_0 and v^*, and let P_0 be the path between v_0 and v^* in T_0.

Claim 3. *If $X \neq X_0$ is a red hyperedge, then either X contains all leaves of T_0, or X contains a node in $V - V(T_0)$.*

Proof. If $X \subseteq V(T_0)$ and it does not contain all leaves, then there is a Steiner tree for X that is smaller than T_0. □

Claim 4. *If $X \subseteq V(P_0)$ is in $\mathcal{E}$, then either X is an edge of P_0, or $|X| \geq 3$.*

Proof. Otherwise there would be a Steiner tree for X_0 that has the same size as T_0 and has a shorter path between two leaves. □

We use the tree T_0 and the path P_0 to define $h(X)$ for every $X \in \mathcal{E}$.

- Let $h(X_0) = v_0$.
- If $X \in \mathcal{E}$ is an edge of T_0, then let $h(X)$ be the node further away from v_0.
- If $X - V(T_0) \neq \emptyset$, then let $h(X)$ be an arbitrary node in $X - V(T_0)$.
- If $X \subseteq V(T_0)$ is a red hyperedge that contains all leaves, then let $h(X) = v_0$.
- If $X \subseteq V(T_0)$ is not an edge of T_0, and X has a blue node $v \notin V(P_0)$, then let $h(X) = v$.
- If $X \subseteq V(T_0)$ is not an edge of T_0, all blue nodes of X are on the path P_0, and X has a red node $v \notin V(P_0)$, then let $h(X) = v$.
- If $X \subseteq V(P_0)$, and $|X| \geq 3$, then let $h(X)$ be the node of X closest to v_0 for which X has another node of the same colour.

To prove the theorem, we consider a set Z for which $Z - h(\mathcal{E}_Z) = \emptyset$, and show some properties which lead to a contradiction. As in the non-bipartite case, it is enough to consider sets with $f(Z) = 1$.

Claim 5. *Suppose that $Z - h(\mathcal{E}_Z) = \emptyset$ for a set $\emptyset \neq Z \subsetneq V$. Then $Z \cap V(P_0)$ consists of the blue nodes of P_0.*

Proof. Since Z contains at least one node of every edge of the path P_0, it suffices to prove that Z does not contain a red node in $V(P_0)$. Let us denote the nodes of the path P_0 by $v_0, v_1, \ldots, v_k = v^*$. First we show that if $v_i \in Z$, $X \in \mathcal{E}_Z$ and $h(X) = v_i$, then either X is an edge of P_0, or $i = 0$ and X is a red hyperedge containing all leaves of T_0. Indeed, the only other possibility is that $X \subseteq V(P_0)$ and $|X| \geq 3$; if that occurs, let us choose such a v_i and X with i maximal. This choice implies that the nodes $v_i, v_{i+1}, \ldots, v_k$ are alternatingly in Z and not in Z. But by the choice of $h(X)$, X contains a node of the same colour as v_i which has higher index, which means that this node is also in Z. Thus $|X \cap Z| \geq 2$, which contradicts $X \in \mathcal{E}_Z$.

If $v_i \in Z$ for some $i > 0$, the above implies that $\{v_{i-1}, v_i\} \in \mathcal{E}_Z$, so $v_{i-1} \notin Z$. This means that Z either contains all blue nodes of P_0 or it contains all red nodes of P_0. But the latter is impossible: it would contain both v_0 and v^* that are both leaves of T_0, thus $|X \cap Z| \geq 2$ would hold for every red set containing all leaves of T_0, so there would be no $X \in \mathcal{E}_Z$ for which $h(X) = v_0$. □

Since $Z \cap X_0 \neq \emptyset$, Z must contain a red node that is not in $V(P_0)$. Let v_r be a red node in Z that is closest to v_0 in T_0. There is a hyperedge $X \in \mathcal{E}_Z$ for which $h(X) = v_r$. We distinguish two cases.

- If X is an edge vv_r of T_0: there is an edge uv on the path from v_0 to v in T_0, where u is red and v is blue. Since $|X \cap Z| = 1$, we know that $v \notin Z$, but this means that $u \in Z$ since $\{u, v\} \cap Z \neq \emptyset$. But this is a contradiction, since u is closer to v_0 than v_r.

- If X is not an edge of T_0: we know that X has a blue node v on the path P_0, otherwise $h(X)$ cannot be v_r. But then $v \in X \cap Z$ and $v_r \in X \cap Z$, so $|X \cap Z| \geq 2$, which contradicts $X \in \mathcal{E}_Z$.

This contradiction proves that the set Z does not exist. □

Remark. In a previous version of the paper, we proposed the following conjecture:

Conjecture 6. *Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph. Suppose that $\mathcal{E}_Z \cap \mathcal{E}_{V-Z} = \emptyset$ for every $\emptyset \neq Z \subsetneq V$. Then there is a function $h : \mathcal{E} \to V$ such that $Z - h(\mathcal{E}_Z) \neq \emptyset$ for every $\emptyset \neq Z \subsetneq V$.*

It turned out that this conjecture is false. The counterexample is the hypergraph $H = (V, \mathcal{E})$ where $|V| = 6$ and $\mathcal{E} = \{123, 345, 561, 246, 14, 25, 36\}$.

References

[1] E. Boros, V. Gurvich, *Perfect graphs are kernel solvable*, Discrete Mathematics 159 (1996), 33–55.

[2] M. Chudnovsky, N. Robertson, P. Seymour, R. Thomas, *The Strong Perfect Graph Theorem*, Annals of Math 164 (2006), 51–229.