A quick proof for the matroidal structure of a source location problem

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Abstract

Based on a general matroid construction of Edmonds, a short proof is described for a result of Nagamochi, Ishii, and Ito asserting that the k-sources in a source location problem form a matroid. The reduction allows us to derive a min-max formula for the minimum cardinality of a k-source.

Let $D = (V, A)$ be a digraph with at least $k+1$ nodes in which there are no parallel edges. For a subset $Z \subseteq V$ and a node $v \in V$, let $\kappa^+(Z, v)$ denote the maximum number of paths from Z to v which are disjoint apart from their joint terminal node v. We call a subset Z a k-**source** if $\kappa^+(Z, v) \geq k$ holds for every node $v \in V - Z$. By this definition V is always a k-source.

The following statement can easily be derived from the directed node version of Menger's theorem.

Proposition 1. *A subset $Z \subseteq V$ is a k-source if and only if $|\Gamma^-(X)| \geq k$ holds for every non-empty subset $X \subseteq V - Z$ where $\Gamma^-(X) := \{u \in V - X :$ there is an edge $uv \in A$ with $v \in X\}$.* •

The problem of finding a minimum cardinality, or more generally, a minimum weight k-source was solved by H. Nagamochi, T. Ishii and H. Ito [2] who proved that that the k-sources form the generators of a matroid. (A subset of the ground-set is called a generator of the matroid if it includes a basis). See also the recent book of Nagamochi and Ibaraki [3] (page 298). Once this reduction is made, the matroid greedy algorithm can be applied since an independence oracle for the matroid in question can be constructed relying on Menger's theorem and on an MFMC routine.

Our proof makes use of the following result of J. Edmonds.

THEOREM 2 (Edmonds, [1]). *Let b be a non-negative, non-decreasing, intersecting submodular set function on a ground-set S, and let*

$$\mathcal{F}_b := \{I \subseteq S : |Y| \leq b(Y) \text{ for every } Y \subseteq I\}. \tag{1}$$

Then $M = (S, \mathcal{F}_b)$ is a matroid whose rank function is given by

$$r_b(Z) = \min\{\textstyle\sum_i b(Z_i) + |Z - \cup_i Z_i| : \{Z_i\} \text{ is a sub-partition of } Z\}. \bullet \tag{2}$$

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The theorem we are going to reprove is as follows.

THEOREM 3 (Nagamochi, Ishii, Ito [2]). *The set $\{I : I = V - Z,\ Z$ a k-source$\}$ satisfies the independence axioms of a matroid. In other words, the k-sources form the generators of a matroid.*

Proof. Let S_k denote the set of nodes with in-degree at least k. Since a k-source contains every node of in-degree smaller than k, the complements of k-sources are subsets of S_k. Define a set function $b_k : 2^{S_k} \rightarrow \mathbf{Z}$ as follows. Let $b_k(\emptyset) = 0$ and

$$b_k(X) := |\Gamma^-(X)| + |X| - k \text{ when } X \neq \emptyset. \tag{3}$$

Lemma 4. *Set function b_k is non-negative, non-decreasing, and intersecting submodular.*

Proof. For any edge $uv \in A$ with $v \in X \subseteq S_k$, the tail u is either in X or in $\Gamma^-(X)$. Since there are at least k edges entering v, we obtain that $|X| + |\Gamma^-(X)| \geq k$ and hence b_k is indeed non-negative.

Let $X \subseteq Y \subseteq S_k$. The tail of an edge entering X is either in $Y - X$ or in $V - Y$. Therefore $|\Gamma^-(X)| \leq |\Gamma^-(Y)| + |Y - X|$ from which $|\Gamma^-(X)| + |X| \leq |\Gamma^-(Y)| + |Y|$ follows, that is, b_k is indeed non-decreasing.

The intersecting submodularity of b_k follows from the well-known fact that the set function $|\Gamma^-(X)|$ is (fully) submodular. •

Apply Theorem 2 to $b := b_k$, and consider the matroid $M_{b_k} = (S_k, \mathcal{F}_{b_k})$ determined by the theorem. A subset $I \subseteq S_k$ is independent in M_{b_k} precisely if $b_k(X) \geq |X|$ holds for every subset X of I, that is, $|\Gamma^-(X)| \geq k$. By Proposition 1, this is just equivalent to requiring that $V - I$ is a k-source. Therefore the independent sets of matroid M_{b_k} are the complements of k-sources, as required. •

The rank formula (2) in Theorem 2 allows one to derive a min-max formula for the minimum cardinality of a k-source.

THEOREM 5. *In a directed graph $D = (V, A)$, the minimum cardinality of a k-source is equal to*

$$|V - S_k| + \max\{\textstyle\sum_i [k - |\Gamma^-(X_i)|] : \{X_1, \dots, X_t\} \text{ a sub-partition of } S_k\} \tag{4}$$

where S_k is the set of nodes with in-degree at least k.

Proof. Consider the matroid M_{b_k} determined on ground-set S_k by function (3). The rank of M_{b_k} is given by formula (2): $\min\{\sum_i b_k(X_i) + |S_k - \cup X_i| : \{X_1, \dots, X_t\}$ a sub-partition of $S_k\}$. By Theorem 3, the minimal k-sources are the complements (with respect to V) of the bases of matroid M_{b_k}. Therefore the minimum cardinality of a k-source equals $|V| - \min\{\sum_i b_k(X_i) + |S_k - \cup X_i| : \{X_1, \dots, X_t\}$ a sub-partitition of $S_k\} = |V - S_k| - \min\{\sum_i [|\Gamma^-(X_i)| - k + |X_i|] - |\cup X_i| : \{X_1, \dots, X_t\}$ a sub-partition of $S_k\} = |V - S_k| + \max\{\sum_i [k - |\Gamma^-(X_i)|] : \{X_1, \dots, X_t\}$ a sub-partition of $S_k\}$. •

References

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[2] H. Nagamochi, T. Ishii, and H. Ito, *Minimum cost source location problem with vertex-connectivity requirements in digraphs*, Information Processing Letters, 80 (2001) 287-294.

[3] H. Nagamochi and T. Ibaraki, Algorithmic aspects of graph connectivities, Cambridge University Press, 2008, Encyclopedia of Mathematics and Its Applications 123. (ISBN-13: 9780521878647)