Well-balanced orientations of mixed graphs

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Abstract

We show that deciding if a mixed graph has a well-balanced orientation is NP-complete.

1 Introduction

The following problem was raised in [3]:

Problem 1. Given a mixed graph, decide whether it has an orientation that is a well-balanced orientation of the underlying undirected graph.

In this note we will prove that this question is NP-complete. In other words: if someone has started to orient the edges of a graph then it is NP-complete to decide whether this can be finished to a well-balanced orientation. Let us give some necessary definitions; our notations will follow those introduced in [2].

Definition 1. A *mixed graph* is determined by the triple (V, E, A) where V is the set of nodes, E is the set of undirected edges and A is the set of directed edges. The *underlying undirected graph* is obtained by deleting the orientation of the arcs in A. An *orientation of a mixed graph* means that we orient the undirected edges (and leave the directed ones).

Definition 2. A *well-balanced orientation* of a graph $G = (V, E)$ is an orientation $\vec{G}$ satisfying

$$\lambda_{\vec{G}}(x, y) \geq \lfloor \lambda_G(x, y)/2 \rfloor \text{ for all } x, y \in V.$$

If furthermore the in-degree and the out-degree of any node differs by at most one in $\vec{G}$ then we call it a *best-balanced orientation.* An orientation $\vec{M}$ of a mixed graph M that is a well-balanced orientation of the underlying undirected graph will simply be called a *well-balanced orientation of M*.

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Theorem 1. It is an NP-complete problem to decide whether a given mixed graph has an orientation that is a well-balanced orientation of the underlying undirected graph.

We note that several related problems were shown to be NP-complete in [1], including the one of deciding if a mixed graph has a *best*-balanced orientation.

For the reduction we need a special form of the VERTEX COVER problem.

Lemma 1. Given a graph with $2n$ vertices and no isolated vertex, it is NP-complete to decide whether there exists a vertex cover of size at most n.

Proof. It is well known that the VERTEX COVER problem is NP-complete, we will reduce it to the above problem. Assume we are given an instance of the VERTEX COVER problem consisting of a graph $G = (V, E)$ and a positive integer k where the question is whether G has a vertex cover of size at most k. We may clearly assume that G has no isolated vertex. Distinguish the following cases:

1. If $k = |V|/2$ then we are done.
2. If $k > |V|/2$ then let G' be the disjoint union of G and $K_{t,1}$ for $t = 2k+1-|V|$. Then G has a vertex cover of size at most k iff G' has a vertex cover of size at most $k+1$. Since G' has $2k+2$ vertices, G' is an instance of the problem in the lemma.
3. If $k < |V|/2$ then let G' be the disjoint union of G and K_t for $t = |V|+2-2k$. Then G has a vertex cover of size at most k iff G' has a vertex cover of size at most $k+t-1$. As G' has $2(k+t-1)$ vertices, G' is again an instance of the problem in the lemma.

The reduction takes clearly polynomial time, hence the lemma follows. □

Proof of Theorem 1. The problem is easily seen to be in NP, so let us prove its completeness. To this end we will reduce the problem in Lemma 1 to our problem using a construction similar to those in [1]. So suppose we are given an instance $G' = (V', E')$ of the problem in Lemma 1. We remark that we wanted to avoid isolated vertices in G' only to make the following argumentation simpler. Consider the following mixed graph $M = (V, E, A)$: the vertex set V will contain two designated vertices s and t, $d_{G'}(v)+2$ vertices $y^v, x_0^v, x_1^v, x_2^v, \dots, x_{d_{G'}(v)}^v$ for every $v \in V'$, and one vertex x_e for every $e \in E'$. Let us fix an ordering of V', say $V' = \{v_1, v_2, \dots, v_n\}$. The arc set A of M contains a directed circuit on $s, x_0^{v_1}, x_0^{v_2} \dots, x_0^{v_n}$ in this order, a pair of oppositely directed arcs between s and y^v for every $v \in V'$, arcs from x_i^v to x_{i+1}^v for every $v \in V'$ and every i between 0 and $d_{G'}(v)-1$, two parallel arcs from x_e to s for every $e \in E'$ and finally for each $v \in V'$ take an arbitrary order of the $d_{G'}(v) = d$ edges of G' incident to v, say $e^1, e^2, \dots, e^d$ and include the arc $(x_i^v, x_{e^{i-1}})$ for any $2 \le i \le d-1$ and the arcs $(x_d^v, x_{e^{d-1}})$ and (x_d^v, x_{e^d}).

The edge set E of M contains one edge between t and y^v and one edge between y^v and x_1^v for every $v \in V'$.

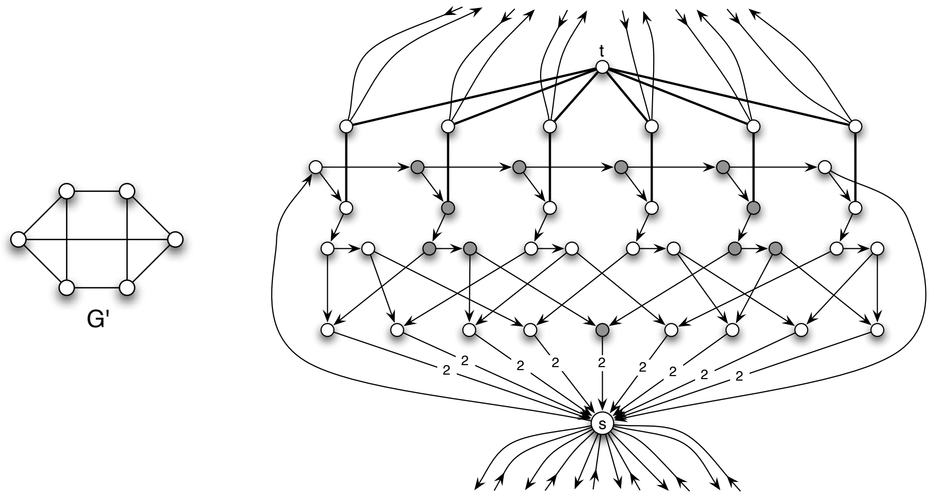

Figure 1: Illustration of the reduction.

The construction is illustrated in Figure 1. The arcs with a label "2" indicate a multiplicity of 2; the undirected edges are drawn in bold.

Let G be the underlying undirected graph of M and $D = (V, A)$ be the directed part of M. Notice that $\lambda_G(x, y) = \min\{d_G(x), d_G(y)\}$ for every $x, y \in V$ (for example one can check that this is true if $y = s$ from which it follows for arbitrary x, y). Observe that $D - t$ is strongly connected and that $\lambda_D(x_e, s) = 2$ for each $e \in E'$.

Observe furthermore that the well-balanced orientations of M are necessarily of the following form: the two edges of E incident to a vertex y^v with $v \in V'$ form a directed path of length two, and for exactly half (i.e. $|V'|/2$) of these, this path starts at t, and for the other half this path ends at t. In other words, $\varrho_{\vec{M}}(y^v) = 2\ \forall v \in V'$ and $\varrho_{\vec{M}}(t) = |V'|/2$ in any well-balanced orientation $\vec{M}$ of M. This is implied by the edge-connectivities in G.

If G' has a vertex cover of size at most $|V'|/2$ then it has one, say S, of size exactly $|V'|/2$. By orienting for every $v \in V'$ the path t, y^v, x_1^v from left to right if $v \in S$, and from right to left otherwise, it is easily seen that we get a well-balanced orientation of M.

Suppose now that M admits a well-balanced orientation $\vec{M}$ and consider the set $S \subseteq V'$ of vertices of G' for which the corresponding directed paths in $\vec{M}$ start at t, that is

$$S := \{v \in V' :\ (t, y^v) \text{ and } (y^v, x_1^v) \text{ are arcs of } \vec{M}\}.$$

We claim that S forms a vertex cover of G': if edge $e = (v_j, v_k) \in E'$ were not covered by S (where $j < k$ are the indices of the vertices in the fixed ordering), then $\varrho_{\vec{M}}(X) = 1$ would contradict the well-balancedness of $\vec{M}$, where

$$\begin{aligned} X &= \{x_e\} \bigcup \{x_0^{v_i} : j \le i \le k\} \\ &\quad \bigcup \{x_i^{v_j} : 1 \le i \le d_{G'}(v_j)\} \bigcup \{x_i^{v_k} : 1 \le i \le d_{G'}(v_k)\} \end{aligned}$$

(the vertices in grey in Figure 1 illustrate this cut). □

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References

[1] Attila Bernáth, *Hardness results for well-balanced orientations*, Tech. Report TR-2006-05, Egerváry Research Group, Budapest, 2006, `www.cs.elte.hu/egres`.

[2] Attila Bernáth, Satoru Iwata, Tamás Király, Zoltán Király, and Zoltán Szigeti, *Recent results on well-balanced orientations*, Tech. Report TR-2006-11, Egerváry Research Group, Budapest, 2006, `www.cs.elte.hu/egres`.

[3] Satoru Iwata, Tamás Király, Zoltán Király, and Zoltán Szigeti, *On well-balanced orientations, counter-examples for related problems*, Tech. Report TR-2004-16, Egerváry Research Group, Budapest, 2004, `www.cs.elte.hu/egres`.