A note on degree-constrained subgraphs

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Abstract

Elementary proofs are presented for two graph theoretic results, originally proved by H. Shirazi and J. Verstraëte using the combinatorial Nullstellensatz.

In an undirected graph $G = (V, E)$ we denote by $d_G(v)$ the degree of $v \in V$. If $F(v) \subseteq \mathbb{N}$ is a set of forbidden degrees for every $v \in V$, then a subgraph $G' = (V, E')$ of G is called **F-avoiding** if $d_{G'}(v) \notin F(v)$ for all $v \in V$.

Theorem 1 (Shirazi, Verstraëte [5]). *If $G = (V, E)$ is an undirected graph and*

$$|F(v)| \leq d_G(v)/2 \textit{ for every node } v, \tag{1}$$

then G has an F-avoiding subgraph.

Theorem 1 appeared first under the name 'Louigi's conjecture' in [1]. A version with $d_G(v)/2$ replaced by $d_G(v)/12$ was given in [1], while $d_G(v)/8$ was proved in [2]. Louigi's conjecture was first settled in the affirmative by H. Shirazi and J. Verstraëte [5]. Their proof is based on the combinatorial Nullstellensatz of N. Alon [3]. Below we give an elementary proof.

Proof. It is well-known that every undirected graph G has an orientation $D = (V, \overrightarrow{E})$ in which

$$\varrho_D(v) \geq \lfloor d_G(v)/2 \rfloor \text{ for every node } v, \tag{2}$$

where $\varrho_D(v)$ denotes the in-degree of v. Indeed, by adding a new node z to G and joining z to every node of G with odd degree, we obtain a graph G^+ in which every degree is even. Hence G^+ decomposes into edge-disjoint circuits and therefore it has an orientation in which the in-degree of every node equals its out-degree. The restriction of this orientation to G satisfies (2). (An orientation with property (2) is also used in [5].) Therefore the following result implies Theorem 1. □

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Theorem 2. *If $G = (V, E)$ is an undirected graph and it has an orientation D for which $\varrho_D(v) \geq |F(v)|$ for every node v, then G has an F-avoiding subgraph.*

Proof. For an undirected edge e, let $\overrightarrow{e}$ denote the corresponding directed edge of D. We use induction on the number of edges. If 0 is not a forbidden degree at any node, then the empty subgraph $(V, \emptyset)$ is F-avoiding. Suppose that $0 \in F(t)$ for a node t. Then $\varrho_D(t) \geq |F(t)| \geq 1$ and hence there is an edge $e = st$ of G for which $\overrightarrow{e}$ is directed toward t. Let $G^- = G - e$ and $D^- = D - \overrightarrow{e}$. Define F^- as follows. Let $F^-(t) = \{i-1 : i \in F(t) \setminus \{0\}\}$, $F^-(s) = \{i-1 : i \in F(s) \setminus \{0\}\}$, and for $z \in V - \{s, t\}$ let $F^-(z) = F(z)$. Since $|F^-(t)| = |F(t)| - 1$, $\varrho_{D^-}(v) \geq |F^-(v)|$ holds for every node v. By induction, there is an F^--avoiding subgraph G'' of G^-. By the construction of F^-, the subgraph $G' := G'' + e$ of G is F-avoiding. □

S. L. Hakimi [4] proved that, given a function $f : V \to \mathbb{Z}_+$, an undirected graph G has an orientation for which $\varrho(v) \geq f(v)$ for every node v if and only if $e_G(X) \geq \sum[f(v) : v \in X]$ holds for every subset $X \subseteq V$ where $e_G(X)$ denotes the number of edges with at least one end-node in X. By combining this with Theorem 2, one obtains the following.

Corollary 3. *If $G = (V, E)$ is an undirected graph and $e_G(X) \geq \sum[|F(v)| : v \in X]$ holds for every subset $X \subseteq V$, then G has an F-avoiding subgraph.*

Along with Theorem 1, the following result was also proved in [5] via the Combinatorial Nullstellensatz. A graph is called **empty** if it has no edges.

Theorem 4 (Shirazi, Verstraëte [5]). *If $G = (V, E)$ is an undirected graph, $0 \notin F(v)$ for all $v \in V$ and $\sum_{v \in V} |F(v)| < |E|$, then G has a nonempty F-avoiding subgraph G'.*

Proof. Again, we use induction on the number of edges. If $d_G(v) \notin F(v)$ for all $v \in V$, then the nonempty $G' = G$ will do. Otherwise there exists a node $t \in V$ where $d_G(t) \in F(t)$. As $0 \notin F(v)$, there is an edge e of G incident to t. Let $G^- = G - e$, let $F^-(t) = F(t) \setminus \{d_G(t)\}$ and for $z \in V - \{t\}$ let $F^-(z) = F(z)$. By induction, there is a nonempty F^--avoiding subgraph G' of G^-. As $d_{G'}(t) < d_G(t)$, this G' is even F-avoiding. □

We remark that Theorems 2 and 4 clearly hold for hypergraphs, as well, with the same proofs. Combining this with the hypergraph variant of Hakimi's theorem, one concludes that also Corollary 3 applies to hypergraphs. However, in Theorem 1 one should replace 2 by the rank of the hypergraph (i.e. by maximum size of a hyperedge). This is already observed by Shirazi and Verstraëte [5]. Note also that both proofs give rise to polynomial algorithms, which were not known before.

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