A note on making two bipartite graphs perfectly matchable

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The following is a special case of a yet unpublished theorem of A. Frank:

Theorem 1 (A. Frank, [1]). *Let $G_1 = (U, V, E_1)$ and $G_2 = (U, V, E_2)$ be two bipartite graphs on the same bipartition (U, V), where $|U| = |V|$. The minimal cardinality of an edge set E such that both $E_1 \cup E$ and $E_2 \cup E$ contain a perfect matching equals the maximum of*

$$\max_{X_1, X_2 \subseteq U,\, X_1 \cap X_2 = \emptyset} \mathrm{def}_1(X_1) + \mathrm{def}_2(X_2)$$

and

$$\max_{Y_1, Y_2 \subseteq V,\, Y_1 \cap Y_2 = \emptyset} \mathrm{def}_1(Y_1) + \mathrm{def}_2(Y_2),$$

where $\mathrm{def}_i(X) = |X| - |\Gamma_i(X)|$ (the deficiency of X), and $\Gamma_i(X)$ is the neighborhood of X in G_i.

The contribution of this note is a new proof for this theorem which reduces it to the matroid intersection theorem. For background on matroids see e.g. Schrijver [2].

Proof. The min $\geq$ max direction is easy.

To prove the other direction we have to find an edge set E of cardinality equal to the maximum above, which has the property that both $E_1 \cup E$ and $E_2 \cup E$ contain a perfect matching.

Let M_U^1 be the dual of the transversal matroid on the ground set U defined by G_1, and let us define M_V^1, M_U^2 and M_V^2 analogously. So if we consider only G_1, the minimal cardinality edge sets with which G_1 contains a perfect matching are exactly the following: we take a base X of M_U^1 and a base Y of M_V^1 and match them arbitrarily.

Let us denote the set of the bases in a matroid M by $\mathcal{B}(M)$.

If $X_1 \in \mathcal{B}(M_U^1)$, $X_2 \in \mathcal{B}(M_U^2)$, $Y_1 \in \mathcal{B}(M_V^1)$ and $Y_2 \in \mathcal{B}(M_V^2)$, then there exists a set of edges of cardinality $\max(|X_1 \cup X_2|, |Y_1 \cup Y_2|)$ with the required property. To find such an edge set we have to match the smaller set of $X_1 \cap X_2$ and $Y_1 \cap Y_2$ into the bigger one, and extend these to a matching between X_1 and Y_1 and between X_2 and Y_2. So the minimum size of the above construction is the maximum of $\min\{|X_1 \cup X_2| : X_1 \in \mathcal{B}(M_U^1), X_2 \in \mathcal{B}(M_U^2)\}$ and $\min\{|Y_1 \cup Y_2| : Y_1 \in \mathcal{B}(M_V^1), Y_2 \in \mathcal{B}(M_V^2)\}$.

Let r_1 and r_2 be the rank functions of the transversal matroids on U defined by G_1 and G_2 respectively. Then $r_1(X) = |X| - \max_{Z \subseteq X} \mathrm{def}_1(Z)$ and $r_2(X) = |X| - \max_{Z \subseteq X} \mathrm{def}_2(Z)$ for every $X \subseteq U$. So with use of the matroid intersection theorem we get

$$
\begin{aligned}
&\min\{|X_1 \cup X_2| : X_1 \in \mathcal{B}(M_U^1),\ X_2 \in \mathcal{B}(M_U^2)\}\\
&\quad = |U| - \max\{|Z_1 \cap Z_2| : U \setminus Z_1 \in \mathcal{B}(M_U^1),\ U \setminus Z_2 \in \mathcal{B}(M_U^2)\}\\
&\quad = |U| - \min\{r_1(U_1) + r_2(U_2) : U_1 \dot\cup U_2 = U\}\\
&\quad = |U| - \min\{|U_1| - \max_{Z_1 \subseteq U_1} \operatorname{def}_1(Z_1) + |U_2| - \max_{Z_2 \subseteq U_2} \operatorname{def}_2(Z_2) : U_1 \dot\cup U_2 = U\}\\
&\quad = \max\{\operatorname{def}_1(Z_1) + \operatorname{def}_2(Z_2) : Z_1, Z_2 \subseteq U,\ Z_1 \cap Z_2 = \emptyset\}.
\end{aligned}
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Similarly the same holds for V and thus the theorem follows. □

References

[1] A. Frank: *personal communication*

[2] A. Schrijver: *Combinatorial Optimization: Polyhedra and Efficiency*, Springer-Verlag, 2003