Matroid intersection for the min-rank oracle

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Abstract

We present a modification of the matroid intersection algorithm for the case when the two matroids are not given explicitly, but only a minimum rank oracle is available. That is for any set we can determine only the minimum of the two ranks.

One may be interested how general is the well-known matroid intersection algorithm? Is it possible to extend it to more abstract frameworks or to some weaker conditions? And a natural example of such weaker conditions is this framework, when we can only ask from the oracle what is the smaller of the two ranks of a given set. This framework allows us to define the problem:

Definition 1. Consider two matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ on a common ground set S. Let r_1 and r_2 be the corresponding rank functions and $r_\star := \min(r_1, r_2)$ (that is $r_\star(X) := \min(r_1(X), r_2(X))$ for every $X \subseteq S$). We will call a set $X \subseteq S$ independent iff it is independent in both matroids, that is $r_\star(X) = |X|$.

With this notation the matroid intersection problem can be formulated as follows: find a subset X of S for which $r_\star(X) = |X|$ and $|X|$ is maximum. It is also easy to formulate the min-max theorem for the matroid intersection in this framework. Let us recall the min-max theorem in the classical form:

Theorem 2 (see [1]). *Maximum size of a common independent set is equal to*

$$\min_{U \subseteq S}(r_1(U) + r_2(S - U)).$$

It is clear that for any independent set F and any subset U of S:

$$|F| = |F \cap U| + |F - U| \le r_\star(U) + r_\star(S - U) \le r_1(U) + r_2(S - U)$$

So using the above theorem we get the following:

Theorem 3. *Maximum size of a common independent set:*

$$\max_{r_\star(F)=|F|} |F| = \min_{U \subseteq S}(r_\star(U) + r_\star(S - U))$$

Thus, the problem of finding a maximum cardinality independent set by using only our restricted oracle lies in NP$\cap$ co-NP. We now present a polynomial time algorithm solving this problem. Our algorithm is a simple extension of the augmenting path algorithm first given by Aigner and Dowling [1971] and Lawler [1975]. For detailed description see [1] Section 41.2.

The algorithm works by starting with some common independent set I (the empty set, for example) and then succeedingly finding a common independent set of bigger cardinality by finding augmenting paths in auxiliary graphs.

Common independent set augmenting algorithm

input: matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ given by rank-minimum oracle $r_\star$, and an independent set I.

output: an independent set I' with $|I'| > |I|$ if exists (positive answer), or set U with $|I \cap U| = r_\star(U)$ and $|I - U| = r_\star(S - U)$ otherwise (negative answer).

Consider three different cases:

Case 1: There exists an $x \in S - I$ st. $r_\star(I + x) > r_\star(I)$. Then $I' = I + x$ is a bigger independent set. Output I'.

Case 2: There are no two elements $x, y \in S - I$ st. $r_\star(I \cup \{x, y\}) > r_\star(I)$. Then it is clear that I is a basis in $\mathcal{M}_1$ or $\mathcal{M}_2$. (Otherwise it could be extended with some x in $\mathcal{M}_1$ and some y in $\mathcal{M}_2$ and $I \cup \{x, y\}$ would have bigger rank in both). That is $|I| = r_\star(S)$. So the answer is negative, output $U = S$.

Case 3: There is no element x for which $r_\star(I + x) > |I|$, but there is some pair of elements x, y for which $r_\star(I+x+y) > |I|$. Denote by $A = \{x \in S-I : r_1(I+x) > |I|\}$ and $B = \{y \in S-I : r_2(I+y) > |I|\}$. It is clear that in this case A and B are disjoint and nonempty. Moreover $r_\star(I + x + y) > |I|$ iff $x \in A$ and $y \in B$ or vice versa. So by querying $r_\star(I + x + y)$ for every $x, y \in S$ we can decide which of the cases apply and determine sets A and B in this case (but not which of the sets is which).

Determining the circuits.

Let $|I| = r_\star(I) = k$. Consider an element $x \in A$. For arbitrary $y \in I$ $r_1(I+x-y) \geq k+1-1 = k$. So $r_\star(I + x - y) < k$ iff y is not an element of the x's fundamental circuit $C_{\mathcal{M}_2}(x, I)$. The similar is true for $x \in B$: $r_\star(I + x - y) < r_\star(I) \iff y \notin C_{\mathcal{M}_1}(x, I)$.

For the elements of $S - I - A - B$ we do the following. We choose an arbitrary element $a \in A$ and $b \in B$. For $x \in S - I - A - B$:

$$r_1(I + a + x) = r_1(I) + 1 = k + 1,$$
$$r_2(I + a + x) = r_2(I) = k.$$

Now, for $y \in I$ we get $r_\star(I + a + x - y) < r_\star(I)$ iff $r_2(I + a + x - y) < k = r_2(I + a + x)$, that is $y \notin C_{\mathcal{M}_2}(a, I) \cup C_{\mathcal{M}_2}(x, I)$. So we can compute this set $C_{\mathcal{M}_2}(x, I) \cup C_{\mathcal{M}_2}(a, I)$ and will call it *the modified fundamental circuit* of x in the second matroid. Similarly, by querying $r_\star(I + b + x - y)$ for all $y \in I$ we can compute $C_{\mathcal{M}_1}(x, I) \cup C_{\mathcal{M}_1}(b, I)$, the modified fundamental circuit in the first matroid.

Auxiliary graph.

Let us construct the auxiliary graph in the same way as in the original matroid intersection algorithm, only using the modified fundamental circuits. For every $x \in S - I - A$ we draw a directed arc from every element of its (modified) fundamental circuit in $\mathcal{M}_1$ to x. And for every $x \in S - I - B$ we put an arc from x to every element of its (modified) fundamental circuit in $\mathcal{M}_2$.

This modified auxiliary graph is as good for the purpose of the algorithm as the original one. The main observation is that a shortest path from A to B cannot use a arc which is not in the original auxiliary graph. Suppose on the contrary, that a path uses an xy arc, where $x \in S - I - A - B$ and $y \in C_{\mathcal{M}_2}(a, I)$ (and not $C_{\mathcal{M}_2}(x, I)$), then this path can be shortened if we start it right with ay arc. Similarly for "false" arcs from $\mathcal{M}_2$.

So with the modified auxiliary graph in the same way as with the original one we can either improve on I, or can show a splitting of S which shows optimality. (Observe for this the set of element reachable from A is the same in the modified auxiliary graph and the original. The argument is similar to that in the previous paragraph.)

References

[1] Alexander Schijver, *Combinatorial Optimization — Polyhedra and Efficiency*, Springer, Berlin, 2003.