Egerváry Research Group
on Combinatorial Optimization

Technical reportS

TR-2008-09. Published by the Egerváry Research Group, Pázmány P. sétány 1/C, H–1117, Budapest, Hungary. Web site: www.cs.elte.hu/egres. ISSN 1587–4451.

Globally Rigid Circuits of the Direction-Length Rigidity Matroid

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May 19, 2008

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Abstract

A two-dimensional mixed framework is a pair (G, p), where $G = (V; D, L)$ is a graph whose edges are labeled as 'direction' or 'length' edges, and p is a map from V to $\mathbb{R}^2$. The label of an edge uv represents a direction or length constraint between $p(u)$ and $p(v)$. The framework (G, p) is called globally rigid if every framework (G, q) in which the direction or length between the endvertices of corresponding edges is the same as in (G, p), can be obtained from (G, p) by a translation and, possibly, a dilation by -1.

We characterize the generically globally rigid mixed frameworks (G, p) for which the edge set of G is a circuit in the associated direction-length rigidity matroid. We show that such a framework is globally rigid if and only if each 2-separation S of G is 'direction balanced', i.e. each 'side' of S contains a direction edge. Our result is based on a new inductive construction for the family of edge-labeled graphs which satisfy these hypotheses. We also settle a related open problem posed by Servatius and Whiteley concerning the inductive construction of circuits in the direction-length rigidity matroid.

1 Introduction

Consider a configuration of points $p_1, p_2, ..., p_n$ in $\mathbb{R}^d$ together with a set of constraints which fix the direction or the length between some pairs p_i, p_j. A basic question is whether the configuration, with the given constraints, is locally or globally unique, up to 'congruence'. Results of this type have applications in CAD [16], localization of sensor networks [4], and in determining molecular conformation [8].

We model the configuration and constraints as a 'mixed framework'. A *mixed graph* $G = (V; D, L)$ is an undirected graph together with a labeling (or bipartition) $D \cup L$ of its edge set. We refer to edges in D as *direction edges* and edges in L as *length edges*. A

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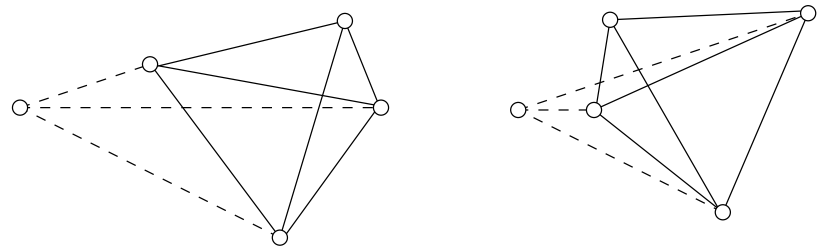

Figure 1: Two equivalent but non-congruent realizations of a mixed graph. We use solid or dashed lines to indicate edges with length or direction labels, respectively.

mixed framework is a pair (G,p), where $G = (V; D, L)$ is a mixed graph and p is a map $p : V \to \mathbb{R}^d$. We say that (G,p) is a *realization* of G in $\mathbb{R}^d$. Two mixed frameworks (G,p) and (G,q) are *mixed-equivalent* (or simply *equivalent*) if (i) $p(u) - p(v)$ is a scalar multiple of $q(u) - q(v)$ for all $uv \in D$ and (ii) $||p(u) - p(v)|| = ||q(u) - q(v)||$ for all $uv \in L$, where $||.||$ denotes the Euclidean norm in $\mathbb{R}^2$. We say that (G,p) is a *length framework* if $D = \emptyset$, is a *direction framework* if $L = \emptyset$, and is a *pure framework* if it is either a length or direction framework. If two pure frameworks satisfy (i) or (ii) then we say that they are direction- or length-equivalent, respectively.

The mixed frameworks (G,p) and (G,q) are *mixed-congruent* (or simply *congruent*) if (i) $p(u)-p(v)$ is a scalar multiple of $q(u)-q(v)$ and (ii) $||p(u)-p(v)|| = ||q(u)-q(v)||$ for all $u, v \in V$. We can define direction-congruence and length-congruence in a similar way for pure frameworks by imposing only (i) or (ii) above.

Note that if $d = 2$ then saying that two mixed frameworks are congruent is equivalent to saying that one can be obtained from the other by a translation and a rotation by 0 or 180 degrees about a point. Similarly, if two pure frameworks are direction-congruent (length-conguent) then one can be obtained from the other by a translation and a dilation (respectively, translation and/or rotation and/or reflection).

The mixed framework (G,p) is *globally mixed-rigid* in $\mathbb{R}^d$ if every framework which is equivalent to (G,p) is congruent to (G,p). *Global direction-rigidity* and *global length-rigidity* of pure frameworks are defined analogously. It is a hard problem to decide if a given length framework is globally length-rigid. Indeed Saxe [15] has shown that this problem is NP-hard even for 1-dimensional frameworks. The problem becomes more tractable, however, if we assume that there are no algebraic dependencies between the coordinates of the points of the framework.

A framework (G,p) is said to be *generic* if the set containing the coordinates of all its points is algebraically independent over the rationals. The characterization of d-dimensional generically globally rigid pure frameworks is known for all d when (G,p) is a direction framework, and for $d \leq 2$ when (G,p) is a length framework. A closely related notion, which plays a key role in these characterizations is rigidity. The mixed framework (G,p) is *mixed-rigid* if there exists an $\epsilon > 0$ such that every mixed framework (G,q) which is equivalent to (G,p) and satisfies $\|p(v) - q(v)\| \leq \epsilon$ for all $v \in V$, is congruent to (G,p). *Direction-* and *length-rigidity* of pure frameworks are defined analogously.

We assume henceforth that $d = 2$ unless specified otherwise. One can develop a rigidity theory for mixed frameworks in much the same way as for pure frameworks. For $(x, y) \in \mathbb{R}^2$ let $(x, y)^\perp = (y, -x)$. The *direction-length rigidity matrix* of a mixed framework (G, p) is the matrix $R(G, p)$ of size $(|D| + |L|) \times 2|V|$, where, for each edge $uv \in D \cup L$, in the row corresponding to uv, the entries in the two columns corresponding to the vertex w are given by: $(p(u) - p(v))^\perp$ if $uv \in D$ and $w = u$; $-(p(u) - p(v))^\perp$ if $uv \in D$ and $w = v$; $(p(u) - p(v))$ if $uv \in L$ and $w = u$; $-(p(u) - p(v))$ if $uv \in L$ and $w = v$; $(0, 0)$ if $w \notin \{u, v\}$. The rigidity matrix of (G, p) defines the *direction-length rigidity matroid* of (G, p) on the ground set $D \cup L$ by linear independence of the rows of the rigidity matrix. The framework is said to be *independent* if the rows of $R(G, p)$ are linearly independent. Any two generic frameworks (G, p) and (G, p') have the same rigidity matroid. We call this the 2-dimensional *direction-length rigidity matroid* $\mathcal{R}(G) = (D \cup L, r)$ of the mixed graph G. We denote the rank of $\mathcal{R}(G)$ by $r(G)$. The mixed graph G is said to be *mixed independent*, or *mixed rigid*, if $r(G) = |D| + |L|$, or $r(G) = 2|V| - 2$, respectively. The following lemma relates this linear algebraic definition of the rigidity of mixed graphs to the previous geometric definition for mixed frameworks.

Lemma 1.1. *[11] Let (G, p) be a mixed framework. If G is mixed rigid then (G, p) is mixed rigid. Furthermore, if (G, p) is generic, then (G, p) is mixed rigid if and only if G is mixed rigid.*

Direction and *length rigidity matrices* and *matroids* can be defined similarly for pure frameworks, as can *direction* and *length independence* and *rigidity* of (unlabelled) graphs, see [17]. Henceforth, we will suppress the prefixes mixed, direction, and length when they are clear from the context.

Length frameworks correspond to the well studied bar-and-joint frameworks, for which the characterization of generic rigidity and generic global rigidity are known up to dimension two. (We refer the reader to [6, 17] for a detailed survey of the rigidity of d-dimensional length frameworks.) A graph is length-rigid in $\mathbb{R}$ if and only if it is connected. The characterization of length-rigid graphs in $\mathbb{R}^2$ is based on the following characterization of length-independent graphs due to Laman. For $G = (V, E)$ a graph and $X \subseteq V$, let $E(X)$ denote the set, and $i(X)$ the number, of edges in $G[X]$, that is, in the subgraph induced by X in G.

Theorem 1.2. *[13] A graph $G = (V, E)$ is length independent if and only if $i(X) \leq 2|X| - 3$ for all $X \subseteq V$ with $|X| \geq 2$.*

Laman's theorem was extended to give a characterization of rigid graphs by Lovász and Yemini [14].

A 1-dimensional generic length-framework (G, p) is globally length-rigid if and only if either G is the complete graph on two vertices or G is 2-connected. The characterization for $d = 2$ is as follows. We say that a graph $G = (V, E)$ is *redundantly length-rigid* if $G - e$ is length-rigid for all edges e of G. The graph G is *k-connected* if $|V| \geq k + 1$ and $G - X$ is connected for all $X \subset V$ with $|X| \leq k - 1$.

Theorem 1.3. *[3, 10] Let (G, p) be a 2-dimensional generic length-framework. Then (G, p) is globally rigid if and only if either G is a complete graph on two or three vertices, or G is 3-connected and redundantly rigid in $\mathbb{R}^2$.*

The linearity of the direction constraints in a direction-framework (G,p) implies that direction-rigidity and global direction-rigidity are equivalent and are determined entirely by the graph G for all direction frameworks (G,p), not just generic frameworks. Direction-independence - and (global) rigidity - were characterized by Whiteley [17]. For the special case of 2-dimensional frameworks, there is a simple transformation which shows that direction-independence and (global) rigidity are equivalent to length-independence and rigidity. In particular, Theorem 1.2 gives

Theorem 1.4. *[17] A graph $G=(V,E)$ is direction independent if and only if $i(X) \leq 2|X|-3$ for all $X \subseteq V$ with $|X| \geq 2$.*

Similarly the above mentioned characterization of length-rigid graphs due to Lovász and Yemini also characterizes (globally) direction-rigid frameworks.

Independent mixed graphs were characterized by Servatius and Whiteley.

Theorem 1.5. *[16] A mixed graph $G=(V;D,L)$ is mixed independent if and only if, for all $X \subseteq V$ with $|X| \geq 2$,*

$$i(X) \leq 2|X|-2 \text{ when } E_D(X) \neq \emptyset \neq E_L(X), \tag{1}$$

and

$$i(X) \leq 2|X|-3 \text{ otherwise.} \tag{2}$$

It is straightforward to use this result to obtain a characterization of rigid mixed graphs. The problem of characterizing when a generic mixed framework (G,p) is globally rigid remains open, however. We give a characterization for globally rigid mixed frameworks (G,p) in which the edge set is a circuit in the direction-length rigidity matroid. This complements the results on generically globally rigid length-frameworks whose edge set is a circuit in the length-rigidity matroid [1], and may serve as a building block to a complete characterization.

1.1 Main results

We first give necessary conditions for global mixed rigidity. We need the following concept. Let G be a 2-connected graph. A 2-*separation* of G is a pair of subgraphs G_1, G_2 such that $G = G_1 \cup G_2$, $|V(G_1) \cap V(G_2)| = 2$ and $V(G_1) - V(G_2) \neq \emptyset \neq V(G_2) - V(G_1)$. When $G=(V;D,L)$ is a mixed graph, we say that a 2-separation (G_1,G_2) is *direction-balanced*, respectively *length-balanced*, if both G_1 and G_2 contain an edge in D, respectively L. We say that (G_1,G_2) is *balanced* if it is both direction-balanced and length-balanced. A 2-separation which is not (direction-, length-) balanced is said to be *(direction-, length-) unbalanced*. We say that G is *(direction-, length-) balanced* if all its 2-separations are (direction-, length-) balanced, see Figure 2.

Lemma 1.6. *Let (G,p) be a generic realization of a mixed graph $G=(V;D,L)$. Suppose that (G,p) is globally rigid. Then*
(a) G is rigid,
(b) G is 2-connected,
(c) G is direction balanced,
(d) G has no non-trivial edge-cut consisting of two direction edges.

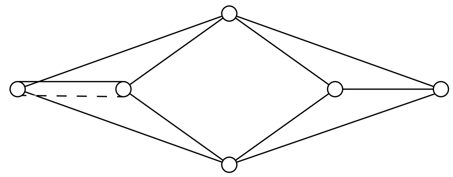

Figure 2: A mixed graph with a direction unbalanced 2-separation.

Proof: The necessity of (a) follows from the definitions of mixed rigidity and global rigidity and Lemma 1.1.

To prove (b) suppose that G has a cut-vertex v and let H be a component of $G-v$. Applying a dilation by -1 centred on $p(v)$ to the points $p(x)$, $x \in V(H)$, gives an equivalent but non-congruent realization of G.

For the proof of (c) let (H_1, H_2) be a direction-unbalanced 2-separation of G, where H_2 is length pure and $V(H_1) \cap V(H_2) = \{u, v\}$. Let (G, q) be the realization of G obtained by reflecting $p(x)$ in the line through $p(u), p(v)$ for each $x \in V(H_2)$. Then (G, q) is equivalent to (G, p) but $||p(x) - p(y)|| \neq ||q(x) - q(y)||$ for all $x \in V(H_2) - \{u, v\}$, $y \in V(H_1) - \{u, v\}$. Thus (G, p) is not globally rigid.

Finally, suppose that $G - \{e, f\}$ has two connected components H_1, H_2 each with at least two vertices, for some $e, f \in D$. Let $e = uv, f = wt$ and let Q be the point of intersection of the lines through $p(u), p(v)$ and $p(w), p(t)$, respectively. Since p is generic, Q exists. Applying a dilation by -1 with center Q to $p(x)$, $x \in V(H_2)$, yields an equivalent but non-congruent realization of G. This proves (d) •

Note that there exist mixed frameworks satisfying all conditions of Lemma 1.6 which are not globally rigid, see Figure 1.

Lemma 1.6(a) implies that mixed rigidity is a necessary condition for global mixed-rigidity. Unlike in length-frameworks, however, redundant mixed rigidity is not a necessary condition for global mixed-rigidity. The fact that rigidity is equivalent to global rigidity for direction frameworks implies that a generic minimally rigid mixed-framework, with exactly one length edge, is globally rigid. Such a mixed framework is clearly not redundantly mixed rigid.

We next describe some sufficient conditions for global mixed rigidity. We use the following operations. A 0-*extension* of a mixed graph $G = (V; D, L)$ adds a new vertex v and new edges vu, vw for vertices $u, w \in V$ with the proviso that, if $u = w$, then the two edges from v to u are of different type. A 1-*extension* (on edge uw and vertex z) for G deletes an edge uw and adds a new vertex v and new edges vu, vw, vz for some vertex $z \in V$, with the provisos that at least one of the new edges has the same type as the deleted edge and, if $z = u$, then the two edges from v to u are of different type. We showed in [11] that 1-extension preserves global rigidity in redundantly rigid generic mixed frameworks. (See [12] for a similar result concerning length frameworks.)

Theorem 1.7. *[11] Let (G, p) and (H, q) be generic mixed frameworks with $|V(H)| \geq 3$. Suppose that (H, q) is globally rigid and that G can be obtained from H by a 1-extension on an edge uw. Suppose further that $H - uw$ is rigid, and $p(x) = q(x)$ for*

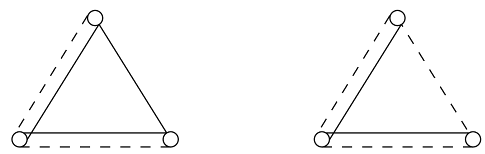

Figure 3: The two mixed circuits on three vertices. These graphs, denoted by K_3^+ and K_3^-, are the smallest (mixed) circuits of the direction-length rigidity matroid.

all $x \in V(H)$. Then (G, p) is globally rigid.

We also showed that a special kind of 0-extension preserves global rigidity.

Theorem 1.8. *[11] Let (G, p) and (H, q) be generic mixed frameworks with $|V(H)| \geq 3$. Suppose that G can be obtained from H by a 0-extension which adds a vertex v incident to two direction edges. Suppose further that $p(x) = q(x)$ for all $x \in V(H)$. Then (G, p) is globally rigid if and only if (H, q) is globally rigid.*

Note that if G is obtained by a 0-extension then G cannot be redundantly mixed rigid.

We will use Theorems 1.7 and 1.8 to show that a special family of generic mixed frameworks are globally rigid. A mixed graph $G = (V; D, L)$ is a *circuit* if $D \cup L$ is a circuit in the direction-length rigidity matroid. The circuit G is a *mixed circuit* if $D \neq \emptyset \neq L$ and otherwise it is a *pure circuit*, see Figure 3. Theorem 1.5 implies that mixed circuits are redundantly rigid mixed graphs with $|D| + |L| = 2|V| - 1$, see Section 3.

We will need another operation on mixed graphs. Suppose that $G_1 = (V_1, E_1)$ and $G_1 = (V_2, E_2)$ are graphs with $V_1 \cap V_2 = \{u, v\}$ and $E_1 \cap E_2 = \{uv\}$. Then we say that the graph $G = (G_1 - uv) \cup (G_2 - uv)$ is a *2-sum of* G_1 *and* G_2, and write $G = G_1 \oplus_2 G_2$. When $G_i = (V_i; D_i, L_i)$ is a mixed graph for each $i \in \{1, 2\}$ and uv has the same type in both G_1 and G_2, their 2-sum is the mixed graph $(V_1 \cup V_2; (D_1 \cup D_2) - \{uv\}, (L_1 \cup L_2) - \{uv\})$.

We first characterize mixed circuits. (This solves an open problem raised by Servatius and Whiteley in [16].)

Theorem 1.9. *Let G be a mixed circuit. Then G can be obtained from K_3^+ or K_3^- by a sequence of 1-extensions and 2-sums with pure K_4's.*

We next obtain a refined characterization for direction balanced mixed circuits.

Theorem 1.10. *Let $G = (V; D, L)$ be a mixed graph. Then G is a direction-balanced mixed circuit if and only if G can be obtained from K_3^+ or K_3^- by 1-extensions and 2-sums with direction-pure K_4's.*

Theorems 1.7 and 1.8 imply that the operations of 1-extension and 2-sum with a direction-pure K_4 preserve global mixed-rigidity. We use this and the fact that K_3^+ and K_3^- are both generically globally rigid to obtain the following characterization of globally rigid mixed circuits.

Theorem 1.11. *Let (G, p) be a generic realization of a mixed circuit. Then (G, p) is globally rigid if and only if G is direction balanced.*

The organization of the paper is as follows. In Section 2 we prove a number of preliminary lemmas on the structure of independent mixed graphs. Mixed circuits are introduced in Section 3. The inductive constructions for mixed circuits and direction balanced mixed circuits are obtained in Sections 4 and 5, respectively. The characterization of globally rigid mixed circuits is deduced in Section 6, while Section 7 contains additional remarks on algorithmic aspects and possible extensions.

We close this section with a characterization of global rigidity for a special kind of d-dimensional generic mixed frameworks.

Theorem 1.12. *Let $G = (V; D, L)$ be a mixed graph in which all pairs of adjacent vertices are connected by both a length and a direction edge, and (G, p) be a d-dimensional generic realization of G. Then (G, p) is globally rigid if and only if G is 2-connected.*

Proof: Necessity follows from (the d-dimensional analogue of) Lemma 1.6(b). To verify sufficiency suppose that G is 2-connected. Let (G, q) be a realization of G which is equivalent to (G, p) and u, v be adjacent vertices of G. By applying a suitable translation and dilation by -1 to (G, q), if necessary, we may suppose that $p(u) = q(u)$ and $p(v) = q(v)$. Let $p = (p_1, p_2, \ldots, p_d)$ and $q = (q_1, q_2, \ldots, q_d)$. Since (G, p) and (G, q) are equivalent, and all pairs of adjacent vertices are connected by both a length and a direction edge, we have $p(x) - p(y) = \pm(q(x) - q(y))$ for all adjacent $x, y \in V$. Hence $p_i(x) - p_i(y) = \pm(q_i(x) - q_i(y))$ for all adjacent $x, y \in V$, and (G, p_i) and (G, q_i) are length-equivalent 1-dimensional length-frameworks. Since G is 2-connected and (G, p_i) is generic, (G, p_i) is globally length-rigid in 1-dimensional space. Since $p_i(u) = q_i(u)$ and $p_i(v) = q_i(v)$, we must have $p_i(x) = q_i(x)$ for all $x \in V$. This holds for all $1 \leq i \leq d$ and hence $p(x) = q(x)$ for all $x \in V$. •

2 Independent graphs and critical sets

Let $G = (V; D, L)$ be an independent mixed graph and $X \subseteq V$ with $|X| \geq 2$. Then X is *mixed critical* if $i(X) = 2|X| - 2$, *direction critical* if $i_D(X) = 2|X| - 3$ and $E_L(X) = \emptyset$, and *length critical* if $i_L(X) = 2|X| - 3$ and $E_D(X) = \emptyset$. We say that X is *pure critical* if X is either direction critical or length critical, and X is *critical* if X is either mixed critical or pure critical.

We shall need the following equalities, which are easy to check by counting the contribution of an edge to each of their two sides.

Lemma 2.1. *Let G be a graph and $X, Y \subseteq V(G)$. Then*

$$i(X) + i(Y) + d(X, Y) = i(X \cup Y) + i(X \cap Y). \tag{3}$$

Lemma 2.2. *Let G be a graph and $X, Y, Z \subseteq V(G)$. Then*

$$\begin{aligned} i(X) + i(Y) + i(Z) + d(X, Y, Z) &= i(X \cup Y \cup Z) + i(X \cap Y) + i(X \cap Z) + \\ &\quad i(Y \cap Z) - i(X \cap Y \cap Z). \end{aligned}$$

Given a graph $G = (V, E)$ and two disjoint subsets $X, Y \subset V$, we use $d(X, Y)$ to denote the number of edges from X to Y. Let $d(X) = d(X, V - X)$. When $X = \{x\}$ we abreviate $d(X)$ to $d(x)$ and refer to $d(x)$ as the *degree* of x.

Lemma 2.3. *Let $G = (V; D, L)$ be an independent mixed graph.*
(a) If X, Y are mixed critical sets with $X \cap Y \neq \emptyset$ then $X \cap Y$ and $X \cup Y$ are both mixed-critical and $d(X, Y) = 0$,
(b) If X, Y are direction (respectively length) critical sets with $|X \cap Y| \geq 2$ then either
(i) $d(X, Y) = 0$ and $X \cap Y$ and $X \cup Y$ are both direction (respectively length) critical, or
(ii) $d(X, Y) = 1$, $X \cup Y$ is mixed critical, and $i_D(X \cup Y) = 2|X \cup Y| - 3$ (respectively $i_L(X \cup Y) = 2|X \cup Y| - 3$) holds.
(c) If X is mixed critical and Y is pure critical with $|X \cap Y| \geq 2$ then $X \cup Y$ is mixed critical, $X \cap Y$ is pure critical and $d(X, Y) = 0$.
(d) If X is length critical and Y is direction critical with $|X \cap Y| \geq 2$ then $X \cup Y$ is mixed critical, $d(X, Y) = 0$, and $|X \cap Y| = 2$.

Proof: The Lemma follows easily from Theorem 1.5 and Lemma 2.1. For example, we may verify (d) as follows.

$$\begin{aligned} 2|X| - 3 + 2|Y| - 3 &= i(X) + i(Y) \\ &= i(X \cap Y) + i(X \cup Y) - d(X, Y) \\ &\leq 2|X \cup Y| - 2 - d(X, Y) \\ &= 2|X| + 2|Y| - 2|X \cap Y| - 2 - d(X, Y), \end{aligned}$$

since $i(X \cap Y) = 0$. Thus $d(X, Y) = 0$, $|X \cap Y| = 2$, and $X \cup Y$ is mixed critical. •

Lemma 2.4. *Let $G = (V; D, L)$ be an independent mixed graph and let X, Y, Z be critical sets satisfying $|X \cap Y| = |Y \cap Z| = |Z \cap X| = 1$ and $X \cap Y \cap Z = \emptyset$.*
(a) If X is mixed critical then Y, Z are both pure critical, $X \cup Y \cup Z$ is mixed critical, and $d(X, Y, Z) = 0$.
(b) If X, Y, Z are direction (respectively length) critical then either
(i) $d(X, Y, Z) = 0$ and $X \cup Y \cup Z$ is direction (respectively length) critical, or
(ii) $d(X, Y, Z) = 1$, $X \cup Y \cup Z$ is mixed critical, and $i_D(X \cup Y \cup Z) = 2|X \cup Y \cup Z| - 3$ (respectively $i_L(X \cup Y \cup Z) = 2|X \cup Y \cup Z| - 3$) holds.

Proof: (a) Since G is independent and the sets X, Y, Z are critical, Theorem 1.5 and Lemma 2.2 imply that $2|X| - 2 + 2|Y| - 3 + 2|Z| - 3 \leq i(X) + i(Y) + i(Z) = i(X \cup Y \cup Z) - d(X \cup Y \cup Z) \leq 2(|X \cup Y \cup Z|) - 2 - d(X \cup Y \cup Z) = 2(|X| + |Y| + |Z| - 3) - 2 - d(X \cup Y \cup Z) = 2|X| - 2 + 2|Y| - 3 + 2|Z| - 3 - d(X \cup Y \cup Z)$. Hence $d(X, Y, Z) = 0$, $X \cup Y \cup Z$ is mixed critical, and Y, Z are both pure critical.

The proof of (b) is similar. •

The following lemma summarizes the connectivity properties of subgraphs induced by critical sets. The definition of a *k-separation* for $k \geq 1$ is analogous to that of a 2-separation given before Lemma 1.6. A graph $G = (V, E)$ is *k-edge-connected* if $G - F$ is connected for all $F \subseteq E$ with $|F| \leq k - 1$.

Lemma 2.5. *Let $G = (V; D, L)$ be an independent mixed graph and let $X \subseteq V$ be a critical set. Then*
(a) $G[X]$ is 2-edge-connected unless X is a pure critical set, $|X| = 2$, and $G[X]$ is an edge.
(b) If (J_1, J_2) is a 1-separation in $G[X]$ then X is mixed critical and $V(J_1)$, $V(J_2)$ are also mixed critical.

Proof: Let $H = G[X]$ and suppose that H can be disconnected by deleting less than two edges. Then there is a set $\emptyset \neq A \subsetneq X$ with $d_H(A) \leq 1$. Hence

$$2|X| - 3 \leq i(X) \leq i(A) + i(X - A) + 1 \leq 2|A| - 2 + 2|X - A| - 2 + 1 = 2|X| - 3.$$

Thus equality must hold everywhere, which implies that X is pure critical and $|A| = 1 = |X - A|$. This proves (a).

Now consider a 1-separation in H and let $V_i = V(J_i)$, $i = 1, 2$. Suppose that X is pure critical. Then

$$2|X| - 3 = i(X) = i(V_1) + i(V_2) \leq 2|V_1| - 3 + 2|V_2| - 3 = 2|V| - 4,$$

a contradiction. Thus X is mixed critical. The previous inequality, when applied to a mixed critical set X, gives that V_i is also mixed critical for $i = 1, 2$. This proves (b). •

3 Circuits in the direction-length rigidity matroid

We can use Theorem 1.5 to determine when a mixed graph is a circuit.

Lemma 3.1. *A mixed graph $G = (V; D, L)$ is a mixed circuit if and only if*
(a) $|D| + |L| = 2|V| - 1$,
(b) $i(X) \leq 2|X| - 2$ for all $X \subset V$ with $2 \leq |X| \leq |V| - 1$ and
(c) $i_D(X) \leq 2|X| - 3$ and $i_L(X) \leq 2|X| - 3$ for all $X \subseteq V$ with $|X| \geq 2$.

Lemma 3.2. *A mixed graph $G = (V; D, L)$ is a pure circuit if and only if*
(a) $|D| + |L| = 2|V| - 2$ and either $D = \emptyset$ or $L = \emptyset$ and
(b) $i(X) \leq 2|X| - 3$ for all $X \subseteq V$ with $2 \leq |X| \leq |V| - 1$.

We say that a pure circuit is a *direction circuit* if $L = \emptyset$ and a *length circuit* if $D = \emptyset$.

It follows that, if G is a circuit, then the graph $\tilde{G}$ obtained from G by interchanging the direction and length edges is also a circuit. In addition, if G is a mixed circuit then $|D| \geq 2$ and $|L| \geq 2$. The smallest mixed circuits, denoted by K_3^+ and K_3^-, are obtained from a cycle on three direction (respectively length) edges by adding two non-parallel length (respectively direction) edges, see Figure 3.

Lemma 3.3. *Let $G = (V; D, L)$ be a mixed circuit. Then G is 3-edge-connected and 2-connected.*

Proof: Consider a bipartition $X \cup Y = V$ $X \cap Y = \emptyset$ of V with $|X|, |Y| \geq 2$. We have $|D \cup L| = i(X) + i(Y) + d(X) \leq 2|X| - 2 + 2|Y| - 2 + d(X) = 2|V| - 4 + d(X) = |E| - 3 + d(X)$. This implies $d(X) \geq 3$. A similar argument shows that G is 2-connected. •

Let $V_3 = \{v \in V : d(v) = 3\}$ denote the set of vertices of degree three in a mixed graph $G = (V; D, L)$. For convenience, vertices of degree three will be called *nodes*. We call $G[V_3]$ the *node subgraph of* G. A node of G with degree at most one (exactly two, exactly three) in the node subgraph of G is called a *leaf node* (*series node*, *branching node*, respectively). A node $v \in V$ is *pure* if all edges incident with v are of the same type. Otherwise v is *mixed*.

Lemma 3.4. *Let $G = (V; D, L)$ be a mixed circuit. Then $G[V_3]$ is a forest.*

Proof: Suppose that C is a cycle in the node subgraph of G. If $V(C) = V(G)$ then each vertex of G is a node. Thus $4|V| - 2 = 2|D \cup L| = 3|V|$, which implies $|V| = 2$ and $|E| = 3$, a contradiction. (Since each circuit on two vertices is pure and has two edges.) So we may assume that $X = V - V(C) \neq \emptyset$. Since each vertex of C is a node of G we have $i(V(C)) + d(V(C)) \leq 2|V(C)|$. Thus $i(X) = 2|V| - 1 - i(V(C)) - d(V(C)) \geq 2|V| - 1 - 2|V(C)| = 2(|V| - |V(C)|) - 1 = 2|X| - 1$, a contradiction. •

Lemma 3.5. *Let $G = (V; D, L)$ be a mixed circuit and let $X \subset V$ be a mixed critical set. Then there is a node of G in $V - X$.*

Proof: Let $Y = V - X$. Since G is 3-edge-connected, we have $d(Y) \geq 3$. Since $i(Y) + d(Y) = |D \cup L| - i(X) = 2|V| - 1 - 2|X| + 2 = 2|Y| + 1$, we obtain

$$\sum_{v \in Y} d(v) = 2i(Y) + d(Y) = 4|Y| + 2 - d(Y) \leq 4|Y| - 1.$$

This implies the lemma. •

It is straightforward to use Lemma 3.1 to deduce the following results on 1-extensions and 2-sums of mixed circuits.

Lemma 3.6. *Let G be a mixed circuit and H be a 1-extension of G. Then H is a mixed circuit.*

Lemma 3.7. *Let G be a mixed graph.*
(a) Suppose G is the 2-sum of two mixed graphs G_1 and G_2. If G_1 is a mixed circuit and G_2 is a pure circuit, then G is a mixed circuit.
(b) Suppose G is a mixed circuit and (H_1, H_2) is a 2-separation of G, where $V(H_1) \cap V(H_2) = \{u, v\}$ and H_2 is pure. Let G_i be obtained from H_i by adding a new edge uv of the same type as the edges of H_2. Then G_1 is a mixed circuit, G_2 is a pure circuit, $G = G_1 \oplus_2 G_2$, $d_G(u) \geq 4$ and $d_G(v) \geq 4$.

Our final result of this section restricts the ways in which two 2-separations in a mixed circuit can 'cross'.

Lemma 3.8. *Let G be a mixed circuit and (H_1, H_2), (H'_1, H'_2) be 2-separations of G. Suppose that H_2 is pure and $V(H'_1) \cap V(H'_2) \not\subseteq V(H_1)$. Then $V(H'_1) \cap V(H'_2) \subseteq V(H_2)$.*

Proof: Suppose the lemma is false. Then $V(H'_1) \cap V(H'_2)$ contains exactly one vertex of $V(H_1) - V(H_2)$ and exactly one vertex of $V(H_2) - V(H_1)$. Let $X_1 = V(H_1)$, $X_2 = V(H_2) \cap V(H'_1)$ and $X_3 = V(H_2) \cap V(H'_2)$. Then $E(G) = E_G(X_1) \cup E_G(X_2) \cup E_G(X_3)$. Since H_2 is pure we have

$$|E(G)| \leq (2|X_1| - 2) + (2|X_2| - 3) + (2|X_3| - 3) = 2|V(G)| - 2.$$

This contradicts the fact that G is a mixed circuit. •

4 Admissible nodes

Let $G = (V; D, L)$ be a mixed graph and $v \in V$ be a node. The *1-reduction operation at v on edges vu, vw* deletes v and all edges incident with v, and adds a new edge uw. (This operation is called *splitting* in [1, 10, 12].) The type of the new edge is arbitrary, unless v is a pure node, in which case the type of uw must be the same as the type of v. The graph obtained by the operation is denoted by G_v^{uw}, or more simply G_v. Note that 1-reduction is the inverse operation to 1-extension. We say that the 1-reduction G_v^{uw} is a *direction 1-reduction* or a *length 1-reduction* according to the type of the new edge uw.

When G is a mixed circuit, a 1-reduction is *admissible* if it results in a smaller mixed circuit. A node v is *admissible* if G has an admissible 1-reduction at v. Otherwise v is *non-admissible*. Examples of non-admissible nodes are given in Figures 5 and 6.

We will determine when a mixed circuit contains an admissible node. We need the following four lemmas. The first characterizes when a 1-reduction at a node v is non-admissible in terms of critical sets containing two neighbours of v. The next three give information on the structure of families of critical sets containing pairs of neighbours v.

Lemma 4.1. *Let G be a mixed circuit and let v be a node in G with edges vu, vw, vt incident to v, $u \neq w$. Suppose that there is no admissible 1-reduction of G at v on vu, vw. Then there exists a mixed critical set X in $G - \{v, t\}$ with $u, w \in X$ or there exists a direction critical set Y and a length critical set Z with $Y \cap Z = \{u, w\}$, $d(Y, Z) = 0$, and $Y \cup Z = V - v$.*

Proof: Let $e_d = uw$ be a direction edge and $e_l = uw$ be a length edge.

Since $G - v + e_d$ is not a mixed circuit there exists either a mixed critical set X in $G - v$ with $\{u, w\} \subseteq X$ and $X \neq V - v$, or a direction critical set $Y \subseteq V - v$ with $\{u, w\} \subseteq Y$. Suppose the first alternative holds. Then $t \notin X$ since otherwise we would have $i(X \cup v) = 2|X + v| - 1$ and $|X + v| \leq |V| - 1$, contradicting the fact that G is a mixed circuit. Thus X would be the required mixed critical set. Hence we may assume that the first alternative does not hold. It follows that there exists a direction critical set $Y \subseteq V - v$ with $\{u, w\} \subseteq Y$. Since $G - v + e_l$ is not a mixed

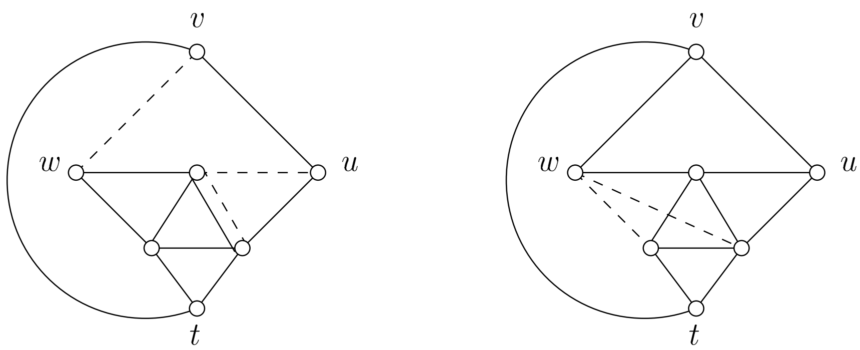

Figure 4: A strong flower and a weak flower on node v.

circuit, there also exists a length critical set Z in $G - v$ with $\{u, w\} \subseteq Z$. Then $|Y \cap Z| \geq 2$ and Lemma 2.3(d) implies that $Y \cup Z$ is mixed critical, $Y \cap Z = \{u, w\}$, and $d(Y, Z) = 0$. Since G is a mixed circuit and $i((Y \cup Z) + v) = 2|(Y \cup Z) + v| - 1$, we have $Y \cup Z = V - v$, as required. •

For a mixed graph $G = (V; D, L)$ and $X \subseteq V$ let $N(X)$ denote the set of *neighbours* of X (that is, $N(X) := \{v \in V - X : uv \in E \text{ for some } u \in X\}$).

Lemma 4.2. *Let $G = (V; D, L)$ be a mixed circuit and v be a node of G with three distinct neighbours u, w and t. Suppose that there exist mixed critical sets X, Y in $G - v$ with $\{u, w\} \subseteq X \subseteq V - \{v, t\}$ and $\{w, t\} \subseteq Y \subseteq V - \{v, u\}$. Suppose further that one of the following conditions hold:*
(i) there exists a mixed critical set Z in $G - v$ with $\{u, t\} \subseteq Z \subseteq V - \{v, w\}$;
(ii) there exists a pure critical set Z in $G - v$ with $\{u, t\} \subseteq Z \subseteq V - \{v\}$.
Then
(a) $X \cup Y = X \cup Z = Y \cup Z = V - v$,
(b) $X \cap Y \cap Z \neq \emptyset$, and
(c) $d(X, Y, Z) = 0$.

Proof: By Lemma 2.3(a), $X \cap Y$ and $X \cup Y$ are both mixed critical sets in $G - v$ and $d(X, Y) = 0$. Since $N(v) \subseteq X \cup Y$, we must have $X \cup Y = V - v$. Since Z is critical, $G[Z]$ is connected by Lemma 2.5. Thus $X \cap Y \cap Z = \emptyset$ would imply $d(X, Y) \geq 1$, contradicting Lemma 2.3(a). Hence $X \cap Y \cap Z \neq \emptyset$. This implies $|X \cap Z|, |Y \cap Z| \geq 2$. Thus Lemma 2.3(a),(c) gives $X \cup Z = V - v$, $Y \cup Z = V - v$, and $d(X, Z) = d(Y, Z) = 0$. Therefore $d(X, Y, Z) = 0$ must also hold. •

A collection of three critical sets X, Y, Z satisfying the hypotheses of Lemma 4.2 with condition (i) (respectively condition (ii)) is called a *strong (respectively weak) flower on node v*, see Figure 4.

Lemma 4.3. *Let $G = (V; D, L)$ be a mixed circuit and v be a pure node of G with three distinct neighbours u, w and t. Suppose that there exists a mixed critical set X and pure critical sets Y, Z of the same type as v in $G - v$ with $\{u, w\} \subseteq X \subseteq V - \{v, t\}$,*

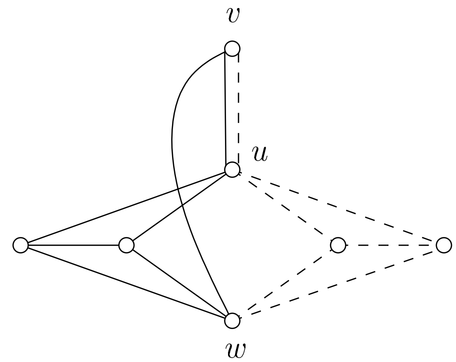

Figure 5: A non-admissible mixed node v.

$\{w,t\} \subseteq Y \subseteq V - \{v\}$, and $\{u,t\} \subseteq Z \subseteq V - \{v\}$. Then there is an unbalanced 2-separation in G.

Proof: First observe that if $|Y \cap Z| \geq 2$ then Lemma 2.3(b) implies that $G[(Y \cup Z)+v]$ contains a pure circuit, a contradiction. Thus $|Y \cap Z| = 1$. Next suppose $|X \cap Y| \geq 2$. Then $X \cup Y$ is mixed critical and $d(X,Y) = 0$ by Lemma 2.3(c). Thus $X \cup Y = V - r$. Since Z is critical, $G[Z]$ is connected by Lemma 2.5. Hence $(Y \cap Z) = \{t\}$ implies $d(X,Y) \geq 1$, a contradiction. So we have $|X \cap Y| = |X \cap Z| = |Y \cap Z| = 1$ and $X \cap Y \cap Z = \emptyset$. Lemma 2.2 now gives that $X \cup Y \cup Z$ is mixed critical and $d(X,Y,Z) = 0$. Since $N(v) \subseteq (X \cup Y \cup Z)$, we must have $X \cup Y \cup Z = V - v$.

Thus $(Y, V - (Y - \{w,t\}))$ is an unbalanced 2-separation, provided $|Y| \geq 3$ holds. Similarly, we have an unbalanced 2-separation when $|Z| \geq 3$. To complete the proof observe that if $|Y| = |Z| = 2$ then, since G is a circuit, we have $|X| \geq 3$. Hence $(X, (Y \cup Z) + v)$ is an unbalanced 2-separation in G. •

Lemma 4.4. *Let $G = (V; D, L)$ be a mixed circuit and v be a pure node of G with three distinct neighbours u, w, t. Then there cannot exist pure critical sets X, Y, Z of the same type as v in $G - v$ with $\{u,w\} \subseteq X \subseteq V - \{v\}$, $\{w,t\} \subseteq Y \subseteq V - \{v\}$, and $\{u,t\} \subseteq Z \subseteq V - \{v\}$.*

Proof: Suppose that the three sets in the lemma do exist. If $|X \cup Y| \geq 2$, say, then Lemma 2.3(b) implies that $G[(X \cup Y) + v]$ contains a pure circuit, a contradiction. So $|X \cap Y| = |X \cap Z| = |Y \cap Z| = 1$ and $X \cap Y \cap Z = \emptyset$. Lemma 2.4(b) now gives that $(X \cup Y \cup Z) \cup \{v\}$ contains a spanning pure circuit, a contradiction. •

Lemma 4.5. *Let $G = (V; D, L)$ be a mixed circuit and v be a mixed node of G. Then exactly one of the following alternatives hold:*
(a) v is admissible;
(b) v has exactly two neighbours u, w and there exists a length critical set X and a direction critical set Y with $X \cap Y = \{u, w\}$, $X \cup Y = V - v$, and $d(X,Y) = 0$;
(c) There is a strong flower on v in G.

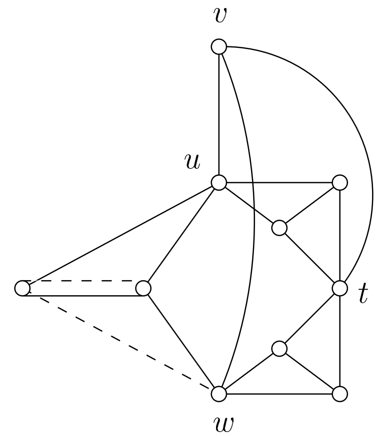

Figure 6: A non-admissible pure node v.

Proof: Assume v is not admissible. If v has only two neighbours then (b) holds by Lemma 4.1. Hence we may suppose that v has three distinct neighbours u, w, t.

Suppose there exists a length critical set X and a direction critical set Y in $G - v$ with $X \cap Y = \{u, w\}$ and $X \cup Y = V - v$. By symmetry we may suppose that $t \in X - Y$. Then all edges incident to t (except possibly vt) are length edges, so t cannot belong to a direction critical set in $G - v$. Since v is not admissible, we must have a mixed critical set Z in $G - v$ with $\{u, t\} \subseteq Z \subseteq V - \{v, w\}$ by Lemma 4.1. Since u is a cutvertex of $G - v - w$, $Z \cap X$ is mixed critical by Lemma 2.5(b). But $D(Z \cap X) = \emptyset$, a contradiction.

Thus, by Lemma 4.1, we must have mixed critical sets X, Y, Z in $G - v$ with $\{u, w\} \subseteq X \subseteq V - \{v, t\}$, $\{w, t\} \subseteq Y \subseteq V - \{v, u\}$, and $\{u, t\} \subseteq Z \subseteq V - \{v, w\}$. The lemma now follows from Lemma 4.2. •

Lemma 4.5 implies the following.

Lemma 4.6. *Let $G = (V; D, L)$ be a mixed circuit with $|V| \geq 4$ and let v be a mixed node of G with $|N(v)| = 2$. If v is non-admissible then there is an unbalanced 2-separation in G.*

We next consider the case when v is a pure node.

Lemma 4.7. *Let G be a mixed circuit and v be a pure node of G. If v is non-admissible then either there is an unbalanced 2-separation in G, or there is a weak or strong flower on v in G.*

Proof: We may suppose that v is non-admissible and, by symmetry, that v is length pure. Since v is pure, we must have $|N(v)| = 3$. Since v is non-admissible there is a mixed critical or length critical set in $G - v$ containing each pair of neighbours of v. The lemma now follows from Lemmas 4.1, 4.2, 4.3, 4.4. •

If v is a node in a mixed circuit G with $N(v) = \{u, w, z\}$ and X is a critical set in $G - v$ with $u, w \in X$ and $v, z \notin X$, then we call X a *v-critical set on u and w*, or simply a *v-critical set*. If $d(z) = 3$ then the 1-reduction G_v^{uw} is non-admissible, since it would make the degree of z be equal to two. In this case $V - \{v, z\}$ is a "trivial" v-critical set on u and w. "Non-trivial" critical sets will be of particular interest: if X is a v-critical set on u and w for some node v with $N(v) = \{u, w, z\}$, and $d(z) \geq 4$, then X is said to be *v-node-critical* or simply *node-critical*.

Lemma 4.8. *Let $G = (V; D, L)$ be a balanced mixed circuit and let $v \in V$ be a node. Let $N(v) = \{x, y, z\}$ with $d(z) \geq 4$, and let X be a mixed v-critical set on x, y. Suppose that either*
(i) there is a non-admissible series node $u \in V - X - v$ with exactly one neighbour w in X, and w is a node, or
(ii) there is a non-admissible leaf node $t \in V - X - v$.
Then there is a mixed node-critical set X^* with $|X^*| > |X|$.

Proof: Since G is balanced, it follows from Lemmas 4.6 and 4.7 that all non-admissible nodes have three distinct neighbours and there exists a (weak or strong) flower on each non-admissible node of G.

Suppose that condition (i) holds. Let $N(u) = \{w, p, q\}$. By our assumption $N(u) \cap X = \{w\}$ and $d(w) = 3$. Since u is a series node, we may assume that $d(p) = 3$ and $d(q) \geq 4$. The non-admissibility of u implies that there exists a (pure or mixed) u-critical set Y on w and p. Since $G[V_3]$ is a forest by Lemma 3.4, we must have $pw \notin D \cup L$ and hence $|Y| \geq 3$. Thus $G[Y]$ has minimum degree at least two by Lemma 2.5(i) and hence Y contains each of the two neighbours of w distinct from u. Since $G[X]$ is connected, at least one of these neighbours of w must belong to X. Thus $|X \cap Y| \geq 2$. By Lemma 2.3 $X^* = X \cup Y$ is a mixed u-critical set on w and p. Since $d(q) \geq 4$ and $p \notin X$, we can also deduce that X^* is a mixed u-node-critical set which properly contains X.

Thus we may assume that condition (ii) holds. We must have $|N(t) \cap X| \leq 2$, since $|N(t) \cap X| = 3$ would imply that $G[X + t]$ contains a circuit. If $|N(t) \cap X| = 2$ then $X + t$ is also mixed v-node-critical and the lemma follows by choosing $X^* = X + t$. Thus we may assume that $|N(t) \cap X| \leq 1$.

Since t is non-admissible, we may choose a flower on t. Since t is a leaf node, this flower contains two t-node-critical sets Y_1 and Y_2 with $Y_1 \cup Y_2 = V - t$ and $d(Y_1, Y_2) = 0$. Furthermore, if one of the neighbours of t is a node, then it belongs to $Y_1 \cap Y_2$.

Suppose that $|X| = 2$. Then, since $G[X]$ is connected and $Y_1 \cup Y_2 = V - t$ and $d(Y_1, Y_2) = 0$, we obtain, without loss of generality, that $X \subseteq Y_1$. If $X = Y_1$ then $|N(t) \cap X| = 2$ would follow, so we must have X properly contained in Y_1. Since X is mixed, Y_1 is also mixed. Hence the lemma follows by choosing $X^* = Y_1$.

Thus we may assume that $|X| \geq 3$. Since $Y_1 \cup Y_2 = V - t$, $t \notin X$, and $|X| \geq 3$, we have $|X \cap Y_1| \geq 2$ or $|X \cap Y_2| \geq 2$. Let us assume, without loss of generality, that $|X \cap Y_1| \geq 2$ holds. By Lemma 2.3, $X \cup Y_1$ is critical. If $(N(t) - Y_1) - X \neq \emptyset$ then the lemma follows by choosing $X^* = X \cup Y_1$, which is a mixed t-node-critical set which properly contains X.

Thus we may assume that $N(t) \cap X = \{s\}$ and $s \notin Y_1$ holds. This implies $d(s) \geq 4$, since if $d(s) = 3$ then we have $s \in Y_1 \cap Y_2$, as noted above. Since $Y_1 \cup Y_2 = V - t$, we have $s \in Y_2$. If $|X \cap Y_2| \geq 2$ then we are done, as above, by choosing $X^* = X \cup Y_2$.

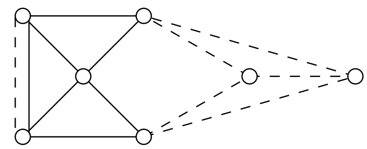

Figure 7: A direction balanced mixed circuit with no admissible nodes.

Thus we may suppose that $X \cap Y_2 = \{s\}$. Since $Y_1 \cup Y_2 = V - t$ and $N(t) \cap Y_1 \cap X = \emptyset$, this implies $|X| < |Y_1|$. Thus if Y_1 is mixed, the lemma follows by choosing $X^* = Y_1$. Otherwise the definition of a flower implies that Y_2 is mixed. Since $X \cap Y_2 \neq \emptyset$, we obtain that $X \cup Y_2$ is mixed critical by Lemma 2.3(a). Thus $X^* = X \cup Y_2$ is mixed t-node-critical set which properly contains X, as required. •

Theorem 4.9. *Let $G = (V; D, L)$ be a balanced mixed circuit with $|V| \geq 4$. Then G has an admissible node.*

Proof: For a contradiction suppose that G is a balanced mixed circuit without admissible nodes. Since G is a circuit, it has at least two nodes. Hence, by Lemma 3.4, the node subgraph of G is a non-empty forest. Let v be a leaf node of G. Since $|V| \geq 4$ and G is balanced, it follows from Lemmas 4.5,4.6, and 4.7 that there exists a flower on v. Hence there exists a mixed v-node-critical set X_v. Choose a maximum size mixed node-critical set X_w with respect to some node w. Since $X_w + w$ is mixed critical, Lemma 3.5 implies that there is a node in $V - w - X_w$. Since $G[V_3]$ is a forest we can deduce that one of the two alternatives of Lemma 4.8 must hold. Thus there is mixed critical node-critical set X^* with $|X^*| > |X_w|$. This contradicts the choice of X_w and completes the proof. •

The mixed circuit in Figure 7 shows that the hypothesis of Theorem 4.9 that G is balanced cannot be weakened to direction (or length) balanced.

We may strengthen Theorem 4.9 by using the following result on the existence of admissible nodes in pure circuits (where a node in a pure circuit G is *admissible* if some 1-reduction of G at v results in a smaller pure circuit).

Theorem 4.10. *[1] Let G be a pure circuit with at least five vertices and x, y, z be vertices of G with xy an edge of G. Then G has an admissible node distinct from x, y, z.*

Theorem 4.11. *Let $G = (V; D, L)$ be a mixed circuit with $|V| \geq 4$. Then either G can be expressed as a 2-sum of a mixed circuit with a pure K_4, or G has an admissible node.*

Proof: Suppose that G has no admissible nodes. By Theorem 4.9 this implies that there is a 2-separation (H_1, H_2) in G for which H_2 is pure. Choose the 2-separation so that H_2 is minimal. Let $V(H_1) \cap V(H_2) = \{a, b\}$. Then H' is a 3-connected pure circuit, where H' is obtained from H_2 by adding an edge ab whose type is the same as that of H_2. If $|V(H')| = 4$ then H' is isomorphic to K_4 are the theorem follows. Suppose that $|V(H')| \geq 5$ holds. Then Theorem 4.10 implies that H' has an admissible node v, different from a, b. Let H'_v obtained from H' by an admissible 1-reduction at v. Since the 2-sum of H_1 and H'_v is a mixed circuit, it follows that v is admissible in G, a contradiction. This completes the proof. •

Theorem 4.11 and Lemmas 3.6 and 3.7 lead to the following inductive construction for mixed circuits, and hence solve an open problem raised by Servatius and Whiteley in [16].

Theorem 4.12. *Let G be a mixed circuit. Then G can be obtained from K_3^+ or K_3^- by a sequence of 1-extensions and 2-sums with pure K_4's.*

We close this section with one more lemma on admissible mixed nodes, which we will need for our characterization of globally rigid circuits.

Lemma 4.13. *Let $G = (V; D, L)$ be a direction balanced mixed circuit and v be a mixed node of G. Suppose that G_v^{xy} is an admissible length 1-reduction at v. Suppose further that G_v^{xy} contains a 2-separation (H_1, H_2) in which H_2 is length pure and $xy \in E(H_2)$. Then there is an admissible direction 1-reduction at v.*

Proof: First suppose v has only two neighbours x, y. Since the 1-reduction on xy is admissible, there is no mixed critical set in $G - v$ containing x, y. Since G is direction balanced, at least one of x or y is in $V(H_2) - V(H_1)$. Hence there is no direction critical containing x, y. Thus G_v^{xy} is also an admissible direction 1-reduction at v.

Now suppose $N(v) = \{u, w, t\}$ and $V(H_1) \cap V(H_2) = \{a, b\}$. Since G is direction balanced, we may assume that $t \in V(H_2) - V(H_1)$. Suppose that the direction 1-reductions on t, u and t, w are both non-admissible. Since all edges incident to t in $G - v$ are length edges there exists no direction critical set in $G - v$ containing t. Thus we must have two mixed critical sets X, Y in $G - v$ with $\{u, t\} \subseteq X$, $X \subseteq V - v - w$, $\{w, t\} \subseteq Y$, $Y \subseteq V - v - u$. Since H_2 is length pure, it follows from Lemma 2.5(ii) that we must have $\{a, b\} \subseteq X \cap Y$. This implies $N(v) \cap \{a, b\} = \emptyset$, so both end-vertices of the edge created by the 1-reduction must be in $V(H_2) - V(H_1)$. By symmetry this implies that we also have a mixed critical set Z in $G - v$ with $\{u, w\} \subseteq Z$, $Z \subseteq V - v - t$, contradicting the assumption that v is admissible. •

5 Feasible nodes

We saw in Lemma 1.6(c) that globally rigid generic mixed frameworks are direction balanced. We shall show in the next section that this necessary condition for global rigidity is also sufficient when the underlying graph is a mixed circuit. Our proof uses induction on the size of the circuit and relies on the recursive construction for direction balanced mixed circuit which we will derive in this section.

A 1-reduction in a direction-balanced mixed circuit G is *feasible* if it results in a smaller direction-balanced mixed circuit. We say that a node v of G is *feasible* if it has a feasible 1-reduction, and otherwise that v is *infeasible*.

Lemma 5.1. *Let v be an admissible node of a direction-balanced mixed circuit G and G_v be the mixed circuit obtained by performing an admissible 1-reduction at v. Suppose that G_v is not direction balanced. Then G_v has a 2-separation (H_1, H_2) such that H_2 is length pure. Furthermore, for every such 2-separation of G_v, $H_2 - H_1$ contains a neighbour of v, and, if v is length pure, then $H_1 - H_2$ also contains a neighbour of v.*

Proof: Since G_v is not direction balanced, G_v has a 2-separation (H_1, H_2) where H_2 is length pure. Since G is direction-balanced, $(H_1 + v, H_2)$ is not a 2-separation of G and hence $H_2 - H_1$ contains a neighbour of v. If v is length pure then $(H_1, H_2 + v)$ is not a 2-separation of G and hence $H_1 - H_2$ contains a neighbour of v. •

Theorem 5.2. *Suppose G is a direction-balanced mixed circuit with at least four vertices. Then either G can be expressed as a 2-sum of a direction-balanced circuit and a direction pure K_4, or G has a feasible node.*

Proof: We proceed by contradiction. Suppose the theorem is false and let G be a counterexample.

Suppose that $G = G_1 \oplus_2 G_2$ for some mixed circuit, G_1, and pure K_4, G_2. Since G is direction-balanced, G_2 must be direction-pure. The fact that G is direction balanced now implies that G_1 is direction-balanced. This contradicts the fact that G is a counterexample. Thus G cannot be expressed as a 2-sum of a mixed circuit and a pure K_4, and hence G has an admissible node by Theorem 4.11.

We say that an admissible 1-reduction of G at a node v is *acceptable* if it is an admissible direction 1-reduction at v if such a splitting exists (and is an admissible length 1-reduction when no admissible direction 1-reduction at v exists). Choose an acceptable 1-reduction $G_w^{x,y}$ of G and a direction unbalanced 2-separation (H_1, H_2) of $G_w^{x,y}$ such that H_2 is length pure and H_2 has as few vertices as possible. Let $V(H_1) \cap V(H_2) = \{u, v\}$ and $N_G(w) = \{x, y, z\}$. Let G_i be obtained by adding a length edge uv to H_i for each $i \in \{1, 2\}$. Using Lemma 3.7 and the minimality of H_2 we have:

Claim 5.3. *G_1 is a mixed circuit and G_2 is a 3-connected length-pure circuit.*

We shall prove that H_2 contains a feasible node of G.

Claim 5.4. *Either $\{x, y, z\} \cap V(H_1 - H_2) \neq \emptyset$ or $\{u, v\} = \{x, y\}$ and $G_w^{x,y}$ is a direction 1-reduction of w onto xy.*

Proof: Suppose the claim is false. Since G is direction balanced, and H_2 is length pure, w must be a mixed node of G and xy must be a length edge of $G_w^{x,y}$. Lemma 4.13 now implies that G has an admissible direction 1-reduction at w which contradicts the fact that $G_w^{x,y}$ is an acceptable 1-reduction of G. •

Claim 5.5. *No node of G_2 in $V(G_2) - \{u, v, x, y, z\}$ is admissible.*

Proof: Suppose $b \in V(G_2) - \{u,v,x,y,z\}$ is an admissable node of G_2. Let $(G_2)_b^{c,d}$ be an admissible 1-reduction of G_2. Then $(G_2)_b^{c,d}$ is a length pure circuit. By Lemma 3.7(a),

$$H = G_1 \oplus_2 (G_2)_b^{c,d} = (G_w^{x,y})_b^{c,d}$$

is a mixed circuit. Since $G_b^{c,d}$ is a 1-extension of H, Lemma 3.6 implies that $G_b^{c,d}$ is a mixed circuit and hence $G_b^{c,d}$ is an admissible 1-reduction in G. Since b is a length-pure node of G, $G_b^{c,d}$ is acceptable. Lemma 5.1 implies that $G_b^{c,d}$ has a 2-separation (H_1', H_2') where H_2' is length-pure and both $H_1' - H_2'$ and $H_2' - H_1'$ contain a neighbour of b. We may suppose that (H_1', H_2') has been chosen such that H_2' is minimal. Let $V(H_1') \cap V(H_2') = \{u', v'\}$. Since u', v' have degree at least four in $G_b^{c,d}$ by Lemma 3.7, they have degree at least four in G. Thus $w \notin \{u', v'\}$.

Since $\{u', v'\}$ is a 2-vertex-cut of $G_b^{c,d}$, it is also a 2-vertex-cut of H. Similarly $\{u, v\}$ is a 2-vertex-cut of H. Since $(G_2)_b^{c,d}$ is a circuit, it is 2-connected by Lemma 3.3. Thus $(G_2)_b^{c,d} - u$ and $(G_2)_b^{c,d} - v$ are both connected. Since $N_G(b) \subseteq V(G_2)$, and since $\{u', v'\}$ separates two of the neighbours of b in $G_b^{c,d}$, we must have either $\{u', v'\} = \{u, v\}$ or $\{u', v'\} \cap (V(G_2) - \{u, v\}) \neq \emptyset$. Applying Lemma 3.8 to H if the latter alternative holds, we have $\{u', v'\} \subseteq V(G_2)$ in both cases. Thus $V(H_1) \subseteq V(H_1')$. If the first alternative of Claim 5.4 holds, then w is adjacent to at least one vertex of $H_1 - H_2$. If the second alternative of Claim 5.4 holds, then w is not a length pure node of G. We may deduce in both cases that $V(H_1)$ and w are both contained in H_1'. This implies that $|V(H_2')| < |V(H_2)|$, which contradicts the minimality of H_2. •

Claim 5.6. *G_2 is isomorphic to K_4.*

Proof: Suppose G_2 is not isomorphic to K_4. By Claim 5.5, all admissible nodes of G_2 are in $\{u,v,x,y,z\}$. Since $uv \in E(G_2)$, Claim 5.4 and Theorem 4.10 imply that $x, y \in V(G_2) - \{u, v\}$, $z \notin V(G_2)$ and u, v, x, y are the only admissible nodes in G_2. Since $G_w^{x,y}$ is an acceptable 1-reduction of G, Lemma 4.13 implies that w is a length-pure node of G. We shall show that x is a feasible node in G.

Since x is an admissible node of G_2, $(G_2)_x^{s,t}$ is a pure circuit for some $s, t \in N_{G_2}(x)$. Let $N_{G_2}(x) = \{q, s, t\}$. Since xy is an edge of G_2 and y is a node of G_2, we must have $y \in \{s, t\}$. Without loss of generality, $y = t$. By Lemma 3.7(a), $H = (G_w^{x,y})_x^{s,y} = G_1 \oplus (G_2)_x^{s,y}$, is a mixed circuit. Since $G_x^{s,w}$ is a 1-extension of H, Lemma 3.6 implies that $G_x^{s,w}$ is a mixed circuit. Thus x is an admissible node of G. Since x is a length-pure node of G, $G_x^{s,w}$ is an acceptable 1-reduction of G. Lemma 5.1 now implies that $G_x^{s,w}$ has a 2-separation (H_1', H_2') where H_2' is length-pure and $H_1' - H_2'$ and $H_2' - H_1'$ both contain a neighbour of x in G. We may suppose that (H_1', H_2') has been chosen such that H_2' is minimal. Let $V(H_1') \cap V(H_2') = \{u', v'\}$. Since u', v' have degree at least four in $G_x^{s,w}$ by Lemma 3.7, they have degree at least four in G. Thus $w \notin \{u', v'\}$.

We proceed as in the proof of Claim 5.5. Since G is direction-balanced, $\{u', v'\}$ separates w and q in $G_x^{s,w}$. Since $\{u', v'\}$ is a 2-vertex-cut of $G_x^{s,w}$, it is also a 2-vertex-cut of H. Similarly $\{u, v\}$ is a 2-vertex-cut of H. Since $(G_2)_x^{s,y}$ is a circuit, it is 2-connected. Thus the graph obtained from $(G_2)_x^{s,y}$ by adding the vertex w and edges sw, sy is 2-connected. Since $N_G(x) \subseteq V(G_2)$, and since $\{u', v'\}$ separates two of the neighbours of x in $G_x^{s,w}$, we must have either $\{u',v'\}=\{u,v\}$ or $\{u',v'\}\cap(V(G_2)-\{u,v\})\neq\emptyset$. Applying Lemma 3.8 to H if the latter alternative holds, we have $\{u',v'\}\subseteq V(G_2)$ in both cases. We may now deduce as in the proof of Claim 5.5 that $|V(H_2)|>|V(H_2')|$. This contradicts the minimality of H_2. •

Claim 5.7. $\{x,y\}\neq\{u,v\}$.

Proof: Suppose $\{x,y\}=\{u,v\}$. Since G is direction balanced, $z\in V(H_2)-\{x,y\}$ and w is not a length pure node of G. Since G_1 and G_2 are circuits and G_2 is length pure, $G_w^{x,y}$ is a direction 1-reduction of w onto xy and xy is a direction edge of H_1. Let $V(H_2)=\{x,y,z,t\}$. Then t is a length-pure node of G.

Let $G_t^{x,y}$ be obtained by performing a 1-reduction t onto a length edge xy. Then $G_t^{x,y}$ can be constructed from G_1 by two 1-extensions. (We first delete the direction edge xy, add the vertex z, length edges zx,zy and a direction edge zx. We then delete the direction edge zx, add w and edges wx,wy,wz of the same type as in G.) Thus $G_t^{x,y}$ is a mixed circuit by Lemma 3.6. Lemma 5.1 now implies that $G_t^{u,v}$ has a 2-separation (H_1',H_2') where H_2' is length-pure and both $H_1'-H_2'$ and $H_2'-H_1'$ contain a neighbour of t. This is impossible since the neighbours of t in G induce a complete graph in $G_t^{x,y}$. Thus $\{x,y\}\neq\{u,v\}$. •

Claim 5.8. $\{x,y\}\subset V(H_2)$ *and* w *is a length-pure node of* G.

Proof: Suppose that $\{x,y\}\not\subset V(H_2)$. Since xy is an edge of $G_w^{x,y}$, we must have $\{x,y\}\subseteq V(H_1)$ and $z\in V(H_2)$. Choose $t\in V(H_2)-\{u,v,z\}$. We have $V(H_2)=\{u,v,z,t\}$ and t is a length-pure node of G. Let $G_t^{u,v}$ be obtained by performing a 1-reduction of t onto a length edge uv. Then $G_t^{u,v}$ can be constructed from G_1 by two 1-extensions so is a mixed circuit by Lemma 3.6. Lemma 5.1 now implies that $G_t^{u,v}$ has a 2-separation (H_1',H_2') where H_2' is length-pure and both $H_1'-H_2'$ and $H_2'-H_1'$ contain a neighbour of t. This is impossible since the neighbours of t in G induce a complete graph in $G_t^{u,v}$. Thus $\{x,y\}\subset V(H_2)$.

We may now use Lemma 4.13, Claim 5.7 and the fact that $G_w^{x,y}$ is an acceptable 1-reduction of G to deduce that w is length-pure. •

Claim 5.9. $\{x,y\}\cap\{u,v\}\neq\emptyset$.

Proof: Suppose the claim is false. Then x and y are both length-pure nodes of G. Let $G_x^{w,v}$ be obtained by performing a 1-reduction of G at x onto a length edge wv. Note that $wv\notin E(G)$ since the neighbour of w distinct from x,y belongs to H_1-H_2. Note further that $G_x^{w,v}$ can be obtained from G_1 by a sequence of two 1-extensions. Thus $G_x^{w,v}$ is a mixed circuit by Lemma 3.6. Lemma 5.1 implies that $G_x^{w,v}$ has a 2-separation (H_1',H_2') where H_2' is length-pure and both $H_1'-H_2'$ and $H_2'-H_1'$ contain neighbours of x. Since each of the neighbours of x in G is a neighbour of y in $G_x^{w,v}$, we must have $y\in V(H_1')\cap V(H_2')$. This contradicts Lemma 3.7(b) since y has degree three in $G_x^{w,t}$. •

We can now complete the proof of the theorem. Using Claims 5.8 and 5.9, and relabelling if necessary, we may suppose that $y = v$ and $V(H_2) = \{u, y, x, t\}$. Thus x and t are length-pure nodes of G. Let $G_x^{w,t}$ be obtained by performing a 1-reduction of G at x onto a length edge wt. Note that $wt \notin E(G)$ since the neighbour of w distinct from x, y belongs to $H_1 - H_2$. Note further that $G_x^{w,t}$ can be obtained from G_1 by a sequence of two 1-extensions. Thus $G_x^{w,t}$ is a mixed circuit by Lemma 3.6. Lemma 5.1 implies that $G_x^{w,t}$ has a 2-separation (H_1', H_2') where H_2' is length-pure and both $H_1' - H_2'$ and $H_2' - H_1'$ contain neighbours of x. Since both the neighbours of x in $G - t$ are neighbours of t in $G_x^{w,t}$, we must have $t \in V(H_1') \cap V(H_2')$. This contradicts Lemma 3.7(b) since t has degree three in $G_x^{w,v}$. •

Theorem 5.10. *Let $G = (V; D, L)$ be a mixed graph. Then G is a direction-balanced mixed circuit if and only if G can be obtained from K_3^+ or K_3^- by 1-extensions and 2-sums with direction-pure K_4's.*

Proof: It is easy to see that the operations of 1-extension and taking a 2-sum with a direction-pure K_4 preserve the property of being a direction-balanced circuit. We may verify the reverse implication by induction on $|V|$ using Theorem 5.2. •

6 Globally rigid circuits

We can now obtain our promised characterization of generically globally rigid mixed circuits. We need one final lemma.

Lemma 6.1. *Suppose G is a mixed graph and $G = G_1 \oplus_2 G_2$ where G_2 is a direction-pure K_4. Let (G, p) be a generic realization of G and p_1 be the restriction of p to G_1. If (G_1, p_1) is globally rigid, then (G, p) is globally rigid.*

Proof: It is straightforward to check that G can be constructed from G_1 by a direction 1-extension and a direction 0-extension. The lemma follows since these operations preserve global rigidity by Theorem 1.7 and Theorem 1.8, respectively. •

Theorem 6.2. *Let (G, p) be a generic realization of a mixed circuit. Then (G, p) is globally rigid if and only if G is direction-balanced.*

Proof: Necessity follows from Lemma 1.6(c). Sufficiency follows from Theorem 5.10 using the facts that both the mixed circuits with three vertices are globally rigid, and that the operations of 1-extension and 2-sum with a direction-pure K_4 preserve global rigidity by Theorem 1.7 and Lemma 6.1, respectively. •

7 Concluding remarks

There exist efficient algorithms to check whether the sparsity conditions (1) or (2) are satisfied (in the underlying unlabeled graph H of G). Condition (1) holds if and only if the edge set of H can be covered by two forests, which can be tested in $O(n^{3/2} \log n^2/m)$ time [5], where n and m denote the number of vertices and edges, respectively. Condition (2) is equivalent to independence in the well-known length rigidity matroid and can be tested in $O(n^2)$ time, see [2] and the references therein. By using these algorithms one can test independence in the mixed rigidity matroid, check whether a given mixed graph G is a mixed (or pure) circuit, and obtain the inductive construction of Theorem 4.12 in polynomial time.

Testing whether G is direction balanced can be done in linear time. This follows by observing that G is direction balanced if and only if all 2-separations (H_1, H_2) in which H_2 is minimal are direction balanced. It is straightforward to obtain these special 2-separations from the cleavage units (or 3-connected components) of H, which can be listed in $O(n+m)$ time [9]. Thus one can also check whether G is a direction balanced mixed circuit and obtain the inductive construction of Theorem 5.10 in polynomial time.

We remark that the results of this paper, together with [11], can be used to characterize the 'globally linked pairs', the 'globally rigid clusters', and the 'uniquely localizable vertices' in mixed circuits, c.f. [12].

7.1 Strongly globally rigid frameworks

Let $G = (V; D, L)$ be a mixed graph and (G, p), (G, q) be 2-dimensional mixed frameworks. Let us say (G, p) and (G, q) are *strongly equivalent* if edges in L have the same length and edges in D have the same 'oriented direction', i.e. $p(u) - p(v) = k(q(u) - q(v))$ for some $k > 0$. We can also define strong rigidity and strong global rigidity. Clearly strong rigidity is the same as rigidity, but this is not true for global rigidity. We can observe that strong global rigidity is not a generic property. For example let (H, p) be strongly globally rigid and let (G, p') be obtained from (H, p) by a 0-extension which adds a vertex v incident with one length edge vu and one direction edge vw. Then the strong global rigidity of (G, p') depends on the ratio of the length of vu and the distance between u and w.

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