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Sparse certificates and removable cycles in l-mixed p-connected graphs

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Abstract

A graph $G = (V, E)$ is called l-mixed p-connected if $G-S-L$ is connected for all pairs S, L with $S \subseteq V$, $L \subseteq E$, and $l|S| + |L| < p$. This notion is a common generalisation of m-vertex-connectivity ($l = 1, p = m$) and m-edge-connectivity ($l \geq m$, $p = m$). If $p = kl$ then we obtain (k, l)-connectivity, introduced earlier by Kaneko and Ota, as a special case.

We show that by using maximum adjacency orderings one can find sparse local certificates for l-mixed p-connectivity in linear time, provided the maximum edge multiplicity is at most l. A by-product of this result is a short proof for the existence of (and a linear time algorithm to find) a cycle C in an l-mixed p-connected graph with minimum degree at least $p + 2$, for which $G - E(C)$ is l-mixed p-connected. This extends a result of Mader on removable cycles in k-vertex-connected graphs.

1 Introduction

We consider undirected graphs which may contain multiple edges but not loops. Let $G = (V, E)$ be a graph and let (A, B, Z) be an ordered partition of V for which $A, B \neq \emptyset$ (but Z may be empty). Then $(E_G(A, B), Z)$ is called a *mixed cut* in G, where $E_G(A, B)$ denotes the set of edges connecting A and B. Let $d_G(X, Y) := |E_G(X, Y)|$. We say that the mixed cut *separates* x and y, for some pair $x, y \in V$, if $x \in A$ and $y \in B$ (or $x \in B$ and $y \in A$) holds. For a positive integer l and for $u, v \in V$ let

$$\mu_l(u, v, G) = \min\{l|Z| + d_G(A, B) : (E_G(A, B), Z) \text{ is a mixed cut separating } u, v \text{ in } G\}$$

be the *local l-mixed connectivity* between u and v in G. We say that G is *l-mixed p-connected* if $|V| \geq \frac{p}{l} + 1$ and $\mu_l(u, v, G) \geq p$ for all pairs $u, v \in V$. By Menger's

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theorem this is equivalent to saying that $G - X$ is $(p - l|X|)$-edge connected for all $X \subset V$.

If $p = kl$, for some positive integer k, then we obtain (k, l)-connectivity (introduced by Kaneko and Ota [10], see also [3]) as a special case. Even this special case is a common generalisation of m-vertex-connectivity (by letting $l = 1, p = k$) and m-edge-connectivity ($l \geq m$, $p = m$).

First we extend results on 'sparse certificates' of vertex- and edge-connectivity to l-mixed connectivity. A spanning subgraph $H = (V, F)$ of a graph $G = (V, E)$ is a *local p-certificate* of G, for some positive integer p, if $\mu(u, v, H) \geq \min\{\mu(u, v, G), p\}$ for all pairs $u, v \in V$. It follows that if G is l-mixed p-connected then every local p-certificate H of G is an l-mixed p-connected spanning subgraph. We say that the certificate H is *sparse* if $|F| = O(pn)$, where $n = |V|$. For vertex- and edge-connectivity it is well-known that sparse local certificates exist, and they can be found in linear time [2, 4, 5, 13]. They can be used to improve the efficiency of algorithms for various connectivity problems. Our proofs will rely on a number of observations made in previous works [5, 6, 8, 14]. In particular, we use 'MA orderings' to find the desired certificates.

Let $d_G(X) = d_G(X, V - X)$ denote the *degree* of X. We simply use $d_G(x)$ if $X = \{x\}$ for some $x \in V$ and omit the subscript G if it is clear from the context. An ordering $\mathcal{V} = (v_1, v_2, \ldots, v_n)$ of V is a *maximum adjacency (MA) ordering* of G if

$$d(V_i, v_{i+1}) \geq d(V_i, v_j) \quad \text{for all } 1 \leq i < j \leq n, \tag{1}$$

where $V_h = \{v_1, v_2, \ldots, v_h\}$ for $1 \leq h \leq n$. An MA ordering of G can be constructed in $O(n + m)$ time [15]. Let $\mathcal{V}$ be a given MA ordering of G. For each vertex v_i, $1 \leq i \leq n$, let the set of edges $E(V_{i-1}, v_i)$ be ordered so that $v_a v_i < v_b v_i$ whenever $a < b$. Let $e_{i,p} \in E(V_{i-1}, v_i)$ denote the p'th edge in this ordering (if it exists). Let $F_j = \{e_{2,j}, e_{3,j}, \ldots, e_{n,j}\}$, $1 \leq j \leq |E|$. We call F_j the *j-th forest of the ordering* in G. Let $G_j = (V, F_1 \cup F_2 \cup \ldots \cup F_j)$ and $\bar{G}_j = G - (F_1 \cup F_2 \cup \ldots \cup F_j)$. An ordering $(v_1, v_2, \ldots, v_n)$ is *continuous*, if for each component X of G there exist integers $1 \leq s \leq t \leq n$ such that $X = \{v_i : s \leq i \leq t\}$, and for all $v_i \in X$ with $i \neq s$ there is an edge $v_i v_j$ with $j < i$. The next lemma follows easily from the previous definitions, see also [6, 14, 15].

Lemma 1.1. *Let $\mathcal{V} = (v_1, v_2, \ldots, v_n)$ be an MA ordering of $G = (V, E)$ and let $F_1, F_2, \ldots, F_{|E|}$ be the forests of $\mathcal{V}$. Then*
(i) $\mathcal{V}$ is continuous,
(ii) (V, F_1) is a maximal spanning forest of G,
(iii) $\mathcal{V}$ is an MA ordering of $G - F_1$.

We call a graph $G = (V, E)$ *l-simple* if $d(u, v) \leq l$ for all $u, v \in V$ (that is, the maximum edge multiplicity in G is at most l). Let $\mathcal{V} = (v_1, v_2, \ldots, v_n)$ be an MA ordering of $G = (V, E)$ and let $F_1, \ldots, F_{|E|}$ be the forests of $\mathcal{V}$ in G.

Lemma 1.2. *Suppose that G is l-simple and let $x, y \in V$, $x \neq y$, belong to the same component of (V, F_p) for some $p \geq 1$. Then $\mu_l(x, y, G_p) \geq p$.*

Proof: We shall prove, by induction on p, that for every mixed cut $(E_{G_p}(A', B'), Z')$ which separates x and y we have $l|Z'| + d_{G_p}(A', B') \geq p$. Since there is a path from x to y in (V, F_p), this inequality is obvious for $p = 1$. So let us suppose that $p \geq 2$, and that the inequality holds up to $p - 1$. Let $G_p^j = G_p - \{F_1, \ldots, F_j\}$. Since $\mathcal{V}$ is an MA-ordering of $\bar{G}_j$ by Lemma 1.1(iii), the p-th forest F_p of $\mathcal{V}$ in G is the $(p - j)$-th forest of $\mathcal{V}$ in $\bar{G}_j$. Thus the edge-set of G_p^j is the union of the first $p - j$ forests of $\mathcal{V}$ in $\bar{G}_j$. Clearly, $\bar{G}_j$ is also l-simple. Thus by induction

$$l|Z'| + d_{G_p^j}(A', B') \geq p - j \tag{2}$$

holds for every mixed cut separating x and y, for all $1 \leq j \leq p - 1$.

Consider a mixed cut $(E_{G_p}(A, B), Z)$. First suppose that $d_{(V,F_1)}(A, B) \geq 1$. Then, by (2), we have $l|Z| + d_{G_p}(A, B) \geq l|Z| + d_{G_p^1}(A, B) + 1 \geq (p-1) + 1 = p$, as required. So we may assume that $E_{(V,F_1)}(A, B) \cap F_1 = \emptyset$.

Since x and y belong to the same component of (V, F_p), it follows from Lemma 1.1(ii),(iii) that x and y belong to the same component of (V, F_h) for all $h \leq p$. Since $E_{(V,F_1)}(A, B) = \emptyset$ and the mixed cut separates x and y, we must have $|Z| \geq 1$. Thus $l|Z| + d_{G_p}(A, B) \geq p$ is obvious for $p \leq l$. Hence we may also assume that $p \geq l + 1$.

Let X be the vertex set of the component of (V, F_1) containing x and y. By relabelling A and B, if necessary, we may suppose that the first vertex of X in $\mathcal{V}$ is not in B. By using Lemma 1.1(i) and $E_{(V,F_1)}(A, B) = \emptyset$, it follows that there is a vertex $v_l \in Z \cap X$ for which $l < j$ for all $v_j \in B \cap X$. Let z be the first vertex of $Z \cap X$ in $\mathcal{V}$.

Claim 1.3. $d_{\bar{G}_l}(z, B) = 0$.

Proof: For a contradiction suppose that $zw \in F_h$ for some $w \in B$ and $h \geq l + 1$. By Lemma 1.1(ii) we have $w \in X$. Since G is l-simple and $h \geq l + 1$, it follows that $zw \notin F_i$ for some $1 \leq i \leq l$. Now the choice of z and $E_{(V,F_1)}(A, B) = \emptyset$ imply that w has no neighbour in G which preceeds z in $\mathcal{V}$. So by the definition of the forests of $\mathcal{V}$ each of the $m \leq l$ copies of edge zw should have been included in one of the first m forests of $\mathcal{V}$, a contradiction. •

Let $Z' = Z - z$ and $A' = A \cup \{z\}$. By using Claim 1.3, $p \geq l + 1$ and (2) we obtain $l|Z| + d_{G_p}(A, B) \geq l|Z| + d_{G_p^l}(A, B) = l + l|Z'| + d_{G_p^l}(A', B) \geq l + (p - l) = p$, as required. •

Theorem 1.4. *Let G be l-simple. Then G_p is a local p-certificate for l-mixed connectivity.*

Proof: Let $q = \mu_l(u, v, G)$ and suppose, for a contradiction, that $\mu_l(u, v, G_p) < \min\{p, q\}$. Then there exists a mixed cut $(E_{G_p}(A, B), Z)$ separating u and v with $l|Z| + d_{G_p}(A, B) < \min\{p, q\}$. Since $l|Z| + d_G(A, B) \geq \mu_l(u, v, G) = q$, there is an edge $e = xy \in E_G(A, B) - E_{G_p}(A, B)$. By Lemma 1.1(ii)(iii) F_p is a maximal forest in $\bar{G}_{p-1}$, thus x and y belong to the same component of (V, F_p). Thus $\mu_l(x, y, G_p) \geq p$

by Lemma 1.2, a contradiction. •

Theorem 1.4 extends earlier results of Nagamochi and Ibaraki [13] (see also [5]) on certificates for vertex- and edge-connectivity. G_p is 'sparse', since $|E(G_p)| \leq p(n-1)$, and it can be found in linear time [13].

We may also deduce extensions of some previous 'extremal' results on vertex- and edge-connectivity. Let $v = v_n$, $p = d_G(v)$, and let u be the last neighbour of v in $\mathcal{V}$. Then Lemma 1.2 implies the following.

Corollary 1.5. *Let $G = (V, E)$ be an l-simple graph. Then there is a pair $u, v \in V$ with $\mu_l(u, v, G) = \min\{d(u), d(v)\}$.*

This generalises a result of Mader [11]. We call $G = (V, E)$ *minimally l-mixed p-connected* if G is l-mixed p-connected but $G - e$ is not l-mixed p-connected for any $e \in E(G)$. It is easy to show that minimally l-mixed p-connected graphs are l-simple, using that $|V| \geq \frac{p}{l} + 1$ (see [10, Lemma 2]). For a local p-certificate H of a minimally l-mixed p-connected graph G we must have $G = H$. Thus Theorem 1.4 implies that a minimally l-mixed p-connected graph $G = (V, E)$ has $|E| \leq p(n-1)$. Similarly, by taking $v = v_n$ for any MA ordering of G, Lemma 1.2 implies that every minimally l-mixed p-connected graph G has a vertex v (in fact, at least two vertices) with $d_G(v) = p$. This was proved earlier for (k, l)-connected graphs by Kaneko and Ota [10].

Theorem 1.4 and Corollary 1.5 do not hold without assuming that the graph is l-simple. To see this let $G = C_n^{2k+1}$, the graph obtained from a cycle of length n by replacing every edge by $2k + 1$ parallel edges. This graph is $2k$-mixed $4k$-connected, but G_{4k} is not. Furthermore, the minimum degree of G equals $4k + 2$, but $\mu_{2k}(u, v, G) \leq 4k + 1$ for all pairs u, v.

Sparse local p-certificates for vertex- or edge-connectivity can also be obtained by taking the union of p 'scan first search' (SFS) forests, see [2, 4] for details. Since the forests of an MA ordering are SFS forests, this leads to a more general result with a somewhat simpler proof in these special cases. However, scan first search forests cannot be applied in the more general setting of l-mixed connectivity, even if the graph is l-simple. Consider $G = (V, E)$ with $V = \{a, b, c, d, e\}$ and $E = \{ab, ab, ac, ac, bc, bc, cd, cd, ce, ce, ed, ed, be\}$. In this graph $F_1 = \{ab, ac, cd, ce\}$, $F_2 = \{ac, bc, cd, de\}$, $F_3 = \{ab, bc, ce, de\}$, $F_4 = \{be\}$ satisfy that F_i is an SFS forest in $G - \cup_{j=1}^{i-1} F_j$ for $1 \leq i \leq 4$. However, although b and e belong to the same component of F_4, we have $\mu_2(b, e, G) < 4$. Thus the natural extension of Lemma 1.2 fails.

2 Removable cycles

A cycle C is a *p-removable cycle* (with respect to l-mixed connectivity) in $G = (V, E)$ if $G - E(C)$ is a local p-certificate. In particular, if G is l-mixed p-connected then $G - E(C)$ is also l-mixed p-connected for a p-removable cycle C. We can use Theorem 1.4 to show that l-simple graphs with high minimum degree contain p-removable cycles.

Theorem 2.1. *Let $G = (V, E)$ be l-simple and suppose that $d_G(v) \geq p + 2$ for all $v \in V - y$ for some designated vertex $y \in V$. Then there is a p-removable cycle in G. Furthermore, a p-removable cycle can be found in linear time.*

Proof: Let $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ be an MA ordering of G with $v_1 = y$ and let G_p be the union of the first p forests of $\mathcal{V}$ in G, as defined earlier. By Theorem 1.4 G_p is a local p-certificate of G with respect to l-mixed connectivity. Since $v_n \neq y$, we have $d_G(v_n) \geq p + 2$ and hence it follows from the definition of the forests of $\mathcal{V}$ that $v_n v_i \in F_{p+1}$ and $v_n v_j \in F_{p+2}$ for some $i \leq j$. By Lemma 1.1(i) there is a path P from v_n to v_j in (V, F_{p+1}). Thus $C = P + v_n v_j$ is a cycle in $(V, F_{p+1} \cup F_{p+2})$. Since the forests of $\mathcal{V}$ are edge-disjoint, $E(C) \cap E(G_p) = \emptyset$. Hence C is a p-removable cycle in G. The running time follows from the fact that an MA ordering can be computed in linear time. •

By taking $l = 1$ and $p = k$ we obtain an old result of Mader [12]: every simple k-vertex-connected graph with minimum degree at least $k+2$ has a cycle C for which $G - E(C)$ is k-vertex-connected. See also [9]. Note that no polynomial algorithm was known to identify a removable cycle, even for the special case of k-vertex-connectivity.

3 Concluding remarks

Our motivation to investigate mixed connectivity and removable cycles came from the problem of finding k-vertex-connected orientations of graphs. A conjecture of Frank [7] states that a graph has a k-vertex-connected orientation if and only if it is 2-mixed $2k$-connected. This conjecture is still open, even for $k = 2$, but we have been able to verify it for Eulerian graphs in the special case $k = 2$. In the proof we needed Theorem 2.1 for $l = 2$ and $p = 4$. See [1] for more details.

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