Egerváry Research Group
on Combinatorial Optimization

Technical reportS

TR-2003-06. Published by the Egrerváry Research Group, Pázmány P. sétány 1/C,
H–1117, Budapest, Hungary. Web site: www.cs.elte.hu/egres . ISSN 1587–4451.

The d-Dimensional Rigidity Matroid of Sparse Graphs

Bill Jackson and Tibor Jordán

September 12, 2003

The d-Dimensional Rigidity Matroid of Sparse Graphs

Bill Jackson⋆ and Tibor Jordán⋆⋆

Abstract

Let $\mathcal{R}_d(G)$ be the d-dimensional rigidity matroid for a graph $G = (V, E)$. For $X \subseteq V$ let $i(X)$ be the number of edges in the subgraph of G induced by X. We derive a min-max formula which determines the rank function in $\mathcal{R}_d(G)$ when G has maximum degree at most $d+2$ and minimum degree at most $d+1$. We also show that if d is even and $i(X) \leq \frac{1}{2}[(d+2)|X| - (2d+2)]$ for all $X \subseteq V$ with $|X| \geq 2$ then E is independent in $\mathcal{R}_d(G)$. We conjecture that the latter result holds for all $d \geq 2$ and prove this for the special case when $d = 3$. We use the independence result for even d to show that if the connectivity of G is sufficiently large in comparison to d then E has large rank in $\mathcal{R}_d(G)$. We use the case $d = 4$ to show that, if G is 10-connected, then G can be made rigid in $\mathbb{R}^3$ by pinning down approximately three quarters of its vertices.

1 Introduction

We shall only consider graphs without loops or multiple edges. A *framework* (G, p) in d-space is a graph $G = (V, E)$ and an embedding $p : V \to \mathbb{R}^d$. The *rigidity matrix* of the framework is the matrix $R(G, p)$ of size $|E| \times d|V|$, where, for each edge $v_iv_j \in E$, in the row corresponding to v_iv_j, the entries in the d columns corresponding to vertex i (j) contain the d coordinates of $(p(v_i) - p(v_j))$ $((p(v_j) - p(v_i))$, respectively), and the remaining entries are zeros. See [7] for more details. The rigidity matrix of (G, p) defines the *rigidity matroid* of (G, p) on the ground set E by independence of rows of the rigidity matrix. A *framework* (G, p) is *generic* if the coordinates of the points $p(v)$, $v \in V$, are algebraically independent over the rationals. Any two generic frameworks (G, p) and (G, p') have the same rigidity matroid. We call this the d-dimensional *rigidity matroid* $\mathcal{R}_d(G) = (E, r_d)$ of the graph G. We denote the rank of $\mathcal{R}_d(G)$ by $r_d(G)$.

⋆School of Mathematical Sciences, Queen Mary, University of London, Mile End Road, London E1 4NS, England. E-mail: `B.Jackson@qmul.ac.uk` Supported by the Royal Society/ Hungarian Academy of Sciences Exchange Programme.

⋆⋆Department of Operations Research, Eötvös University, Pázmány Péter sétány 1/C, 1117 Budapest, Hungary. E-mail: `jordan@cs.elte.hu` Supported by the Hungarian Scientific Research Fund grant no. T037547, F034930, and FKFP grant no. 0143/2001.

Lemma 1.1. *[7, Lema 11.1.3] Let (G, p) be a framework in $\mathbb{R}^d$. Then rank $R(G, p) \leq S(n, d)$, where $n = |V(G)|$ and*

$$S(n,d) = \begin{cases} nd - \binom{d+1}{2} & \text{if } n \geq d+1 \\ \binom{n}{2} & \text{if } n \leq d+1. \end{cases}$$

We say that a graph $G = (V, E)$ is *rigid* in $\mathbb{R}^d$ if $r_d(G) = S(n, d)$. (This definition is motivated by the fact that if G is rigid and (G, p) is a generic framework on G, then every smooth deformation of (G, p) which preserves the edge lengths $||p(u) - p(v)||$ for all $uv \in E$, must preserve the distances $||p(w) - p(x)||$ for all $w, x \in V$, see [7].) We say that G is *M-independent*, *M-dependent* or an *M-circuit* in $\mathbb{R}^d$ if E is independent, dependent or a circuit, repectively, in $\mathcal{R}_d(G)$. For $X \subseteq V$, let $E_G(X)$ denote the set, and $i_G(X)$ the number, of edges in $G[X]$, that is, in the subgraph induced by X in G. We use $E(X)$ or $i(X)$ when the graph G is clear from the context. A *cover* of G is a collection $\mathcal{X}$ of subsets of V, each of size at least two, such that $\cup_{X \in \mathcal{X}} E(X) = E$.

Lemma 1.1 implies the following necessary condition for G to be M-independent.

Lemma 1.2. *If $G = (V, E)$ is M-independent in $\mathbb{R}^d$ then $i(X) \leq S(|X|, d)$ for all $X \subseteq V$.*

It also gives the following upper bound on the rank function.

Lemma 1.3. *If $G = (V, E)$ is a graph then*

$$r_d(G) \leq \min_{\mathcal{X}} \sum_{X \in \mathcal{X}} S(|X|, d)$$

where the minimum is taken over all covers $\mathcal{X}$ of G.

The converse of Lemma 1.2 also holds for $d = 1, 2$. The case $d = 1$ follows from the fact that the 1-dimensional rigidity matroid of G is the same as the cycle matroid of G, see [1, Theorem 2.1.1]. The case $d = 2$ is a result of Laman [2]. Similarly, the inequality given in Lemma 1.3 holds with equality when $d = 1, 2$. The case $d = 2$ is a result of Lovász and Yemini [4]. Neither of these statements hold for $d \geq 3$. Indeed, it remains an open problem to find good characterizations for independence or, more generally, the rank function in the d-dimensional rigidity matroid of a graph when $d \geq 3$.

We show in Section 3 that the converse of Lemma 1.2 holds and that equality holds in Lemma 1.3 for all d in the special case when G is connected and has maximum degree at most $d + 2$ and minimum degree at most $d + 1$. In addition we show in Section 4 that if we strengthen the necessary condition for M-indendence given in Lemma 1.2 to $i(X) \leq (\frac{d}{2} + 1)|X| - (d + 1)$ then it becomes sufficient to imply that G is M-independent in $\mathbb{R}^d$ for all even $d \geq 2$. We conjecture that the latter result holds for all $d \geq 2$ and prove this for the special case when $d = 3$ in Section 5. In Section 6 we use the result from Section 4 to show that a highly connected graph G has large rank in $\mathcal{R}_d(G)$. We use the case $d = 4$ in Section 7 to show that, if G is 10-connected, then G can be fixed in $\mathbb{R}^3$ by pinning down roughly three quarters of its vertices.

2 Preliminary lemmas

We need the following results. The first three lemmas appear in [7].

Lemma 2.1. *[7, Lemma 11.1.9] Suppose $G = G_1 \cup G_2$.*
(a) If $|V(G_1) \cap V(G_2)| \geq d$ and G_1, G_2 are rigid in $\mathbb{R}^d$ then G is rigid in $\mathbb{R}^d$.
(b) If $|V(G_1) \cap V(G_2)| \leq 1$ and G_1, G_2 are M-independent in $\mathbb{R}^d$ then G is M-independent in $\mathbb{R}^d$.

Lemma 2.2. *[7, Lemma 11.1.1] Let $G = (V, E)$ be a graph and $v \in V$ with $d(v) \leq d$. Then G is M-independent in $\mathbb{R}^d$ if and only if $G - v$ is M-independent in $\mathbb{R}^d$.*

Lemmas 2.2 and 1.2 immediately imply the following elementary result.

Lemma 2.3. *Let G be a graph on at most $d + 2$ vertices. If $G \neq K_{d+2}$ then G is M-independent in $\mathbb{R}^d$. If $G = K_{d+2}$ then G is an M-circuit in $\mathbb{R}^d$.*

Let v be a vertex in a graph G. Suppose $w, x \in N(v)$ and $wx \notin E(G)$. We denote the graph $(G - v) + wx$ by G_v^{wx} and say that G_v^{wx} has been obtained by a *splitting of G at v along wx*.

Lemma 2.4. *[7, Theorem 11.1.7] Let v be a vertex of degree $d + 1$ in a graph G. Suppose $w, x \in N(v)$ and $wx \notin E(G)$. If G_v^{wx} is M-independent in $\mathbb{R}^d$ then G is M-independent in $\mathbb{R}^d$. Furthermore, if G is M-independent in $\mathbb{R}^d$, then G_v^{yz} is M-independent in $\mathbb{R}^d$ for some pair $y, z \in N(v)$.*

The next lemma is folklore. We give a proof for the sake of completeness.

Lemma 2.5. *Let $G = (V, E)$ be a graph.*
(a) If G is rigid in $\mathbb{R}^d$ then G is either d-connected or complete.
(b) If G is an M-circuit in $\mathbb{R}^d$ then G is 2-connected and $(d + 1)$-edge-connected.

Proof: (a) Suppose G is not complete and not d-connected. Let $|V| = n$. If $n \leq d+1$ then, since G is not complete, $r_d(G) \leq |E| < \binom{n}{2} = S(n, d)$. Hence G is not rigid. Thus we may suppose that $n \geq d + 2$. Since G is not d-connected, we can find subgraphs $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$ such that $G = G_1 \cup G_2$, $|V_1 \cap V_2| = d - 1$ and $|V_1|, |V_2| \geq d$. Since adding edges to G cannot decrease $r_d(G)$, we may suppose that $G_1 \cap G_2 = K_{d-1}$. By Lemma 2.3, $G_1 \cap G_2$ is M-independent. Let B be a basis for $\mathcal{R}_d(G)$ containing $E(G_1 \cap G_2)$, and $B_i = B \cap E_i$ for $i \in \{1, 2\}$. Let $|V_1| = n_1, |V_2| = n_2$. Using Lemma 1.1, we have

$$\begin{aligned} r_d(G) = |B| = |B_1| + |B_2| - \binom{d-1}{2} &\leq S(n_1, d) + S(n_2, d) - \binom{d-1}{2} \\ &= S(n, d) - 1. \end{aligned}$$

Thus G is not rigid.
(b) The first part of (b) follows from Lemma 2.1(b). To verify the second part of (b), we proceed as follows. Let S be an edge cut in G and (G, p) be a generic framework

in $\mathbb{R}^d$. Since G is an M-circuit, there exists a nowhere zero self stress for G, see [7, page 235]. Thus there exists $\alpha : E \to \mathbb{R} - \{0\}$ such that for all $v \in V$ we have

$$\sum_{u \in N(v)} \alpha(uv)(p(v) - p(u)) = \mathbf{0}.$$

This implies (by conservation of flow) that

$$\sum_{uv \in S} \alpha(uv)(p(v) - p(u)) = \mathbf{0}. \qquad (1)$$

Since (G, p) is generic, we may use (1) to deduce that $|S| \geq d + 1$. •

Let $G = (V, E)$ be a graph. For $X, Y, Z \subseteq V$, let $d(X, Y) = |E(X \cup Y) - (E(X) \cup E(Y))|$ and $d(X, Y, Z) = |E(X \cup Y \cup Z) - (E(X) \cup E(Y) \cup E(Z))|$. We shall need the following equalities, which are easy to check by counting the contribution of an edge to each of the two sides.

Lemma 2.6. *Let G be a graph and $X, Y \subseteq V(G)$. Then*

$$i(X) + i(Y) + d(X, Y) = i(X \cup Y) + i(X \cap Y).$$

Lemma 2.7. *Let G be a graph and $X, Y, Z \subseteq V(G)$. Then*

$$i(X) + i(Y) + i(Z) + d(X, Y, Z) =$$

$$= i(X \cup Y \cup Z) + i(X \cap Y) + i(X \cap Z) + i(Y \cap Z) - i(X \cap Y \cap Z).$$

3 Graphs of maximum degree at most $d + 2$

Let $G = (V, E)$ be a graph and $d \geq 3$ be a fixed integer. We denote the maximum and minimum degrees of G by $\Delta(G)$ and $\delta(G)$, respectively. We say that G is *Laman* if $i(X) \leq S(|X|, d)$ for all $X \subseteq V$. Thus G is Laman if $i(X) \leq d|X| - \binom{d+1}{2}$ for all $X \subseteq V$ with $|X| \geq d + 2$. A set $X \subseteq V$ is *critical* if $|X| \geq 2$ and $i(X) = S(|X|, d)$. Thus X is critical if either $2 \leq |X| \leq d + 1$ and $G[X]$ is complete, or $|X| \geq d + 2$ and $i(X) = d|X| - \binom{d+1}{2}$. Note that it follows from this definition that critical sets X with $|X| = d, d + 1$ also satisfy $i(X) = d|X| - \binom{d+1}{2}$. Let $v \in V$ with $d(v) = d + 1$. A splitting of v along two neighbours u, w in a Laman graph G is *admissible* if the resulting graph G_v^{uw} is also Laman. Otherwise, it is *non-admissible*. The following characterisation of (non-)admissible splits is straightforward.

Lemma 3.1. *A splitting of v along u, w is not admissible in G if and only if there exists a critical set X with $u, w \in X \subset V - v$.*

We shall also need the following elementary properties of critical sets in Laman graphs.

Lemma 3.2. *Let G be a Laman graph and $v \in X \subset V$ with $|X| \geq d$ and $i(X) = d|X| - \binom{d+1}{2}$.*
(a) $G[X]$ is connected.
(b) If $G[X]$ is not complete then $d_X(v) \geq d$ for all $v \in X$.

Proof: (a) Suppose $H = G[X]$ is not connected. Then there exists non-empty subgraphs H_1, H_2 of H such that $H = H_1 \cup H_2$ and $H_1 \cap H_2 = \emptyset$. The fact that $|E(H)| = i(X) = d|X| - \binom{d+1}{2}$ now implies that either H_1 or H_2 is not Laman and contradicts the fact that G is Laman.
(b) Since $G[X]$ is not complete we have $|X| \geq d+2$. Suppose $d_X(v) \leq d-1$ for some $v \in X$. Then $i(X-v) \geq d|X-v| - \binom{d+1}{2} + 1$. This contradicts the fact that G is Laman. •

Lemma 3.3. *Let $G = (V, E)$ be a Laman graph, v be a vertex of degree $d+1$ in G, and $V' = \{x \in N(v) : d_G(v) \geq d+3\}$. Suppose that $G[V']$ is a (possibly empty) complete graph. Then G has an admissible split at v.*

Proof: Arguing by contradiction we suppose that G is a counterexample to the lemma. Let $N(v) = \{v_1, v_2, \ldots, v_{d+1}\}$ and suppose that no split at v is admissible. By Lemma 3.1, we can find a family $\mathcal{F}$ of maximal critical subsets of V such that for each $1 \leq i < j \leq d+1$, there exists $X \in \mathcal{F}$ with $v_i, v_j \in X \subset V - v$. We may suppose that $\mathcal{F}$ has been chosen such that $|\mathcal{F}|$ is minimal. If $|\mathcal{F}| = 1$ then we have $\mathcal{F} = \{X\}$, $N(v) \subseteq X$, and $i(X+v) = i(X) + d + 1 = d|X+v| - \binom{d+1}{2} + 1$. This contradicts the fact that G is Laman. Hence $|\mathcal{F}| \geq 2$. If $G[N(v)]$ were complete then $N(v)$ would be critical and we could take $\mathcal{F} = \{N(v)\}$, contradicting the minimality of $|\mathcal{F}|$. Thus $G[N(v)]$ is not complete. Relabelling if necessary, and using the fact that $G[V']$ is complete, we may assume that $v_1v_2 \notin E$ and $d_G(v_1) \leq d+2$. Choose $X_1 \in \mathcal{F}$ with $v_1, v_2 \in X_1$. Since $G[X_1]$ is not complete, $d_{X_1}(v_1) \geq d$ by Lemma 3.2(b).

Claim 3.4. *If $X_i, X_j \in \mathcal{F}$, $x \in N(v) \cap X_i \cap X_j$ and $G[X_i], G[X_j]$ are not complete, then $d_G(x) \geq d+3$.*

Proof: We have $d_{X_t}(x) \geq d$ for $t \in \{i, j\}$ by Lemma 3.2(b). Also $|X_i \cap X_j| \leq d-1$ by Lemma 2.6 and the maximality of X_i. Since $vx \in E$ we have

$$d_G(x) \geq d_{X_i}(x) + d_{X_j}(x) - (|X_1 \cap X_2| - 1) + 1 \geq d+3.$$

•

Since $|\mathcal{F}| \geq 2$, $N(v) \not\subseteq X_1$, so there exists a vertex $v_j \in N(v) - X_1$. Choose $v_i \in X_1$ with $d_G(v_i) \leq d+2$, for example $v_i = v_1$. There exists $X_{i,j} \in \mathcal{F}$ with $v_i, v_j \in X_{i,j}$. Since $G[X_1]$ is not complete, $v_i \in X_1 \cap X_{i,j}$, and $d_G(v_i) \leq d+2$, we may use Claim 3.4 to deduce that $G[X_{i,j}]$ is complete. Thus $v_iv_j \in E$ for all $v_j \in N(v) - X_1$. In particular,

$$d+2 \geq d_G(v_i) \geq d_{X_1}(v_i) + |N(v) - X_1| + 1 \geq d + |N(v) - X_1| + 1$$

and hence $|N(v) - X_1| = 1$. Relabelling we may assume that $N(v) - X_1 = \{v_{d+1}\}$ and $N(v) \cap X_1 = \{v_1, v_2, \dots, v_d\}$. If $v_{d+1}v_j \in E$ for all $1 \le j \le d$ then $X_1 + v_{d+1}$ would be a critical set in G contradicting the maximality of X_1. Hence we may assume that $v_d v_{d+1} \notin E$. Since $v_i v_{d+1} \in E$ whenever $d_G(v_i) \le d + 2$, we have $d_G(v_d) \ge d + 3$. Since $G[V']$ is complete, this implies that $d_G(v_{d+1}) \le d + 2$. Choose $X_2 \in \mathcal{F}$ with $v_d, v_{d+1} \in X_2$. Then $G[X_2]$ is not complete so $d_{X_2}(v_{d+1}) \ge d$. Since $d_G(v_{d+1}) \le d + 2$ and $G[X_2]$ is not complete, Claim 3.4 implies that $v_{d+1}v_i \in E$ for all $v_i \in X_1 - X_2$. The facts that $d_{X_2}(v_{d+1}) \ge d$ and $d_G(v_{d+1}) \le d + 2$ now give $|X_1 - X_2| = 1$. Hence $|X_1 \cap X_2| = d - 1$. Since $d(X_1, X_2) = 1$, Lemma 2.6 now implies that $X_1 \cup X_2$ is critical in G. This contradicts the maximality of X_1. •

The following example shows that Lemma 3.3 becomes false if we allow $G[V']$ to contain two non-adjacent vertices of degree greater than $d+2$. Let $G = G' - xy$ where $G' = G_1 \cup G_2$, $V(G_1) \cap V(G_2) = \{x, y\}$, and G_i is a complete graph on $d + 2$ vertices for each $i \in \{1, 2\}$. Then G is Laman and has no admissible split at any vertex of degree $d + 1$.

Theorem 3.5. *Let G be a connected graph with $\Delta(G) \le d + 2$ and $\delta(G) \le d + 1$. Then G is M-independent if and only if G is Laman.*

Proof: Necessity follows from Lemma 1.2. To prove sufficiency, we proceed by induction on $|V|$. Since all graphs on at most $d + 1$ vertices are M-independent by Lemma 2.5(b), we may assume $|V| \ge d + 2$. Let v be a vertex of minimum degree in G.

Suppose $G - v$ is disconnected. Since G is connected and $\Delta(G) \le d + 2$, each component $H_i = (V_i, E_i)$ of $G - v$ satisfies the hypotheses of the theorem, and hence is M-independent by induction. Since $d_{H_i+v}(v) \le d$, $G[V_i + v]$ is M-independent by Lemma 2.2. Hence G is M-independent by Lemma 2.1(b).

Thus we may assume $G - v$ is connected. If $d(v) \le d$ then $G - v$ satisfies the hypotheses of the theorem. Hence $G - v$ is M-independent by induction and G is M-independent by Lemma 2.2. Thus we may also assume that $d(v) = d + 1$. By Lemma 3.3, there is an admissible split G_v of G at v. Since $G - v$ is connected, G_v is connected. Since $\Delta(G) \le d + 2$, we have $\Delta(G_v) \le d + 2$ and $\delta(G_v) \le d + 1$. By induction G_v is M-independent. Thus G is M-independent by Lemma 2.4. •

Using Theorem 3.5 and Lemma 2.5(b) we may deduce:

Corollary 3.6. *Let $G = (V, E)$ be a graph with $\Delta(G) \le d + 2$ and $\delta(G) \le d + 1$. Then G is an M-circuit if and only if G is 2-connected, $|E| = d|V| - \binom{d+1}{2} + 1$, and $i(X) \le d|X| - \binom{d+1}{2}$ for all $X \subseteq V$ with $d + 2 \le |X| \le |V| - 1$.*

Corollary 3.7. *Let $G = (V, E)$ be a graph with $\Delta(G) \le d + 2$ and $\delta(G) \le d + 1$. If G is an M-circuit then $G - e$ is rigid for all $e \in E$.*

Corollary 3.8. *Let G be a connected M-independent graph with $\Delta(G) \le d + 2$ and $\delta(G) \le d+1$. Let X_1, X_2 be maximal critical subsets of V and suppose that $|X_i| \ge d+2$ for each $i \in \{1, 2\}$. Then $X_1 \cap X_2 = \emptyset$.*

Proof: Suppose $X_1 \cap X_2 \neq \emptyset$ and choose $x \in X_1 \cap X_2$. Since G is M-independent and X_i is critical, Theorem 3.5 implies that $H_i = G[X_i]$ is rigid for each $i \in \{1,2\}$. By Lemma 2.1(a), $d_{X_i}(x) \geq d$. Thus $d+2 \geq d_G(x) = d_{X_1}(x) + d_{X_2}(x) - d_{X_1 \cap X_2}(x) \geq d + d - d_{X_1 \cap X_2}(x)$ and $d_{X_1 \cap X_2}(x) \geq d-2$. Hence $|X_1 \cap X_2| \geq d-1$.

We first consider the case when $|X_1 \cap X_2| = d-1$. Then $G[X_1 \cap X_2]$ is complete and $d_{X_i}(x) = d$ for each $i \in \{1,2\}$ and all $x \in X_1 \cap X_2$. Since $d \geq 3$ we may choose $y \in X_1 \cap X_2 - \{x\}$. Since $G[X_1]$ is rigid $G[X_1] - x$ is rigid by Lemma 2.2. This contradicts Lemma 2.1(a) since $|X_1 - x| \geq d+1$ and $d_{X_1 - x}(y) = d-1$.

Hence $|X_1 \cap X_2| \geq d$. Lemma 2.6 and Theorem 3.5 now imply that $X_1 \cup X_2$ is critical, contradicting the maximality of X_1, X_2. •

We next use Theorem 3.5 to determine the rank function for graphs of low degree. Let $G = (V,E)$ be a graph and $\mathcal{X}$ be a cover of G. For $X \subseteq V$ let $f(X) = S(|X|, d)$ and $val(\mathcal{X}) = \sum_{X \in \mathcal{X}} f(X)$. We say that $\mathcal{X}$ is 1-*thin* if $|X_i \cap X_j| \leq 1$ for all distinct $X_i, X_j \in \mathcal{X}$.

Theorem 3.9. *Let $G = (V,E)$ be a connected graph with $\Delta(G) \leq d+2$ and $\delta(G) \leq d+1$. Then $r(E) = \min_{\mathcal{X}} val(\mathcal{X})$ where the minimum is taken over all 1-thin covers $\mathcal{X}$ of G.*

Proof: We have $r(E) \leq val(\mathcal{X})$ for all covers $\mathcal{X}$ of G by Lemma 1.3 so it only remains to show that there exists a 1-thin cover $\mathcal{X}$ of G with $r(E) = val(\mathcal{X})$. Let B be a basis for $\mathcal{R}(G)$, $H = (V,B)$,

$$\mathcal{X}_0 = \{X \subseteq V \ : \ X \text{ is a maximal critical set in } H \text{ and } |X| \geq d+2\},$$

$$\mathcal{X}_1 = \{\{u,v\} \ : \ uv \in B \text{ and } uv \notin E_G(X) \text{ for all } X \in \mathcal{X}_0\}$$

and $\mathcal{X} = \mathcal{X}_0 \cup \mathcal{X}_1$. Then $\mathcal{X}$ is 1-thin by Corollary 3.8.

Since each edge of H belongs to a critical subgraph of H, $\mathcal{X}$ covers H. To see that $\mathcal{X}$ covers G, let $e \in E - B$. Then $e \in E(C) \subseteq B + e$ for a unique M-circuit C. Since C is a subgraph of G, $C - e$ is rigid by Corollary 3.7 and $|V(C)| \geq d+2$ by Lemma 2.5(b). Thus $V(C) \subseteq X$ for some $X \in \mathcal{X}_0$ and $e \in E_G(X)$.

We complete the proof by showing that $val(\mathcal{X}) = r(E)$. Let $B_i = B \cap E_H(X_i)$ for each $X_i \in \mathcal{X}$. Since X_i is critical in H we have $|B_i| = f(X_i)$. Since $\mathcal{X}$ is 1-thin the sets B_i are pairwise disjoint and hence

$$r(E) = |B| = \sum_{X_i \in \mathcal{X}} |B_i| = val(\mathcal{X}).$$

•

The graph $G = K_{d+2,d+2}$ shows that Theorems 3.5 and 3.9 become false when $d \geq 4$ if we remove the hypothesis that $\delta(G) \leq d+1$. It is Laman and is an M-circuit, see [7, Example 11.2.4]. Thus it is not M-independent. Furthermore $val(\mathcal{X}) \geq |E|$ for each 1-thin cover $\mathcal{X}$ of G (and $r(E) = |E| - 1$).

Similarly, Theorems Theorems 3.5 and 3.9 become false when $d \geq 4$ if we remove the hypothesis that G is connected since we take G to be the disjoint union of $G = K_{d+2,d+2}$

and an arbitrary M-independent graph of low degree. It is conceivable however that these results remain valid without the hypotheses that $\delta(G) \leq d+1$ and G is connected in the special case when $d = 3$.

Conjecture 3.10. *Let G be a graph with $\Delta(G) \leq 5$. Then G is M-independent in $\mathbb{R}^3$ if and only if G is Laman.*

Conjecture 3.11. *Let G be a graph with $\Delta(G) \leq 5$. Then $r_3(E) = \min_{\mathcal{X}} val(\mathcal{X})$ where the minimum is taken over all 1-thin covers $\mathcal{X}$ of G.*

Note that by Theorems 3.5 and 3.9, it would suffice to prove the above conjectures for 5-regular graphs.

Remark Let $G = (V, E)$ be a graph and $d \geq 1$ be an integer. For $E' \subseteq E$, we say that E' is *L-independent* if either $E' = \emptyset$, or $E' \neq \emptyset$ and the subgraph of G induced by E' is Laman. This definition of independence gives the rigidity matroid of G when $d \leq 2$. We can show that the definition also gives a matroid, $\mathcal{L}_d(G)$, when $d \geq 3$ and $\Delta(G) \leq d + 2$. The rank function of $\mathcal{L}_d(G)$ is $\tilde{r}_d(E') = \min_{\mathcal{X}} val(\mathcal{X})$, where the minimum is taken over all 1-thin covers $\mathcal{X}$ of the subgraph of G induced by E'. Theorem 3.9 shows that $\mathcal{L}_d(G) = \mathcal{R}_d(G)$ when G is connected, $\Delta(G) \leq d + 2$ and $\delta(G) \leq d+1$. Conjectures 3.10 and 3.11 assert that $\mathcal{L}_3(G) = \mathcal{R}_3(G)$ when $\Delta(G) \leq 5$. This equality does not hold in general since $\mathcal{L}_d(K_{d+2,d+2}) \neq \mathcal{R}_d(K_{d+2,d+2})$ when $d \geq 4$.

4 Sparse graphs

Let $G = (V, E)$ be a graph and k be a positive integer. We say that a subset S of E is *independent* if $|S'| \leq k|V(S')| - (2k - 1)$ for all $\emptyset \neq S' \subseteq S$, or equivalently, if

$$i_{G[S']}(X) \leq k|X| - (2k - 1)$$

for all $S' \subseteq S$ and all $X \subseteq V(S')$ with $|X| \geq 2$. It follows from the theory of submodular functions (see [7, Appendix]) that this definition of independence gives rise to a matroid $\mathcal{N}_k(G)$ with ground set E, for every k. It also follows that the rank of E in $\mathcal{N}_k(G)$ can be expressed as follows (see [4] for the special case $k = 2$). Let $\bar{r}_k$ denote the rank function of $\mathcal{N}_k$. Then

$$\bar{r}_k(E) = \min_{\mathcal{X}} \left\{ \sum_{X \in \mathcal{X}} (k|X| - (2k - 1)) \right\} \tag{2}$$

where the minimum is taken over all collections $\mathcal{X} = \{X_1, X_2, ..., X_t\}$ of subsets of V for which $\{E(X_1), E(X_2), ..., E(X_t)\}$ partitions E. (In fact, it suffices to minimize over 1-thin covers of G.)

Suppose that E is independent in $\mathcal{N}_k(G)$. Let v be a vertex of G and G_v be obtained by a splitting of G at v. We say that this splitting is *feasible* if $E(G_v)$ is independent in $\mathcal{N}_k(G_v)$.

Lemma 4.1. *Let $G = (V, E)$ be a graph and v be a vertex of G of degree $2k - 1$. Suppose that E is independent in $\mathcal{N}_k(G)$. Then some splitting of G at v is feasible.*

Proof: Suppose that G has no feasible splitting at v. Let $S = \{wx : w, x \in N_G(v)\}$, $H = (V, E \cup S)$, $\mathcal{N} = \mathcal{N}_k(H)$ and r be the rank function in $\mathcal{N}$. Let $G - v = G' = (V - v, E')$. Since E is independent in $\mathcal{N}$ and $E' \subseteq E$, E' is independent in $\mathcal{N}$. If $r(E' + wx) = r(E') + 1$ for some $wx \in S$ then $E' + wx$ would be independent in $\mathcal{N}$ and G_v^{wx} would be a feasible splitting of G at v. Thus $r(E' + wx) = r(E')$ for all $wx \in S$ and hence $r(E' \cup S) = r(E')$. Let $T = E - E'$. Since $H[S] = K_{2k-1}$, S is independent in $\mathcal{N}$. Thus S can be extended to a basis B of $\mathcal{N}$. Since $H[T \cup S] = K_{2k}$, $T \cup S$ is dependent in $\mathcal{N}$. Thus B contains at most $|T| - 1 = 2k - 2$ edges in T. It follows that

$$r(E \cup S) = r(E' \cup S \cup T) \leq r(E' \cup S) + 2k - 2 = r(E') + 2k - 2 = r(E) - 1.$$

This contradicts the fact that r is monotone. •

Theorem 4.2. *Let $G = (V, E)$ be a graph and let d be an even integer. If E is independent in $\mathcal{N}_{\frac{d}{2}+1}(G)$ then E is independent in $\mathcal{R}_d(G)$.*

Proof: Suppose the theorem is false and let G be a smallest counterexample. Let v be a vertex of minimum degree in G. Since E is independent in $\mathcal{N}_{\frac{d}{2}+1}(G)$, we have $|E| \leq (\frac{d}{2} + 1)|V| - (d + 1)$ and hence $d(v) \leq d + 1$.

Suppose $d(v) \leq d$. Let $G - v = G' = (V - v, E')$. Since E is independent in $\mathcal{N}_{\frac{d}{2}+1}(G)$, E' is independent in $\mathcal{N}_{\frac{d}{2}+1}(G')$. By induction E' is independent in $\mathcal{R}_d(G')$. Using Lemma 2.2, we deduce that E is independent in $\mathcal{R}_d(G)$.

Hence we may assume that $d(v) = d + 1$. By Lemma 4.1, G has a feasible split $G_v = (V - v, E'')$ at v. Then E'' is independent in $\mathcal{N}_{\frac{d}{2}+1}(G_v)$. By induction E'' is independent in $\mathcal{R}_d(G_v)$. Using Lemma 2.4, we deduce that E is independent in $\mathcal{R}_d(G)$. •

Using the definition of independence in $\mathcal{R}(G)$ we immediately obtain:

Corollary 4.3. *Let $G = (V, E)$ be a graph and let d be an even integer. If $i(X) \leq (\frac{d}{2} + 1)|X| - (d + 1)$ for all $X \subseteq V$ with $|X| \geq 2$ then G is M-independent in $\mathbb{R}^d$.*

Note that the case $d = 2$ corresponds to (the difficult part of) Laman's theorem. We believe that Corollary 4.3 is valid for odd values of d as well.

Conjecture 4.4. *Let $G = (V, E)$ be a graph. If $i(X) \leq (\frac{d}{2} + 1)|X| - (d + 1)$ for all $X \subseteq V$ with $|X| \geq 2$ then G is M-independent in $\mathbb{R}^d$.*

5 Sparse graphs in 3-space

In this section we prove (a slightly stronger form of) Conjecture 4.4 for $d = 3$. Throughout this section M-independent refers to independence in $\mathcal{R}_3(G)$.

Theorem 5.1. *Let $G = (V, E)$ be a graph. If*

$$i(X) \leq \frac{1}{2}(5|X| - 7) \tag{3}$$

for all $X \subseteq V$ with $|X| \geq 2$ then G is M-independent.

Proof: We use induction on $|V|$. The theorem holds if $|V| = 2$ since K_2 is M-independent. Hence suppose $|V| \geq 3$. Let v be a vertex of minimum degree in G. Since $|E| = i(V) \leq (5|V| - 7)/2$, v has degree at most four. If $d(v) \leq 3$ then we may apply induction to deduce that $G - v$ is M-independent. Then G is M-independent by Lemma 2.2. Hence we may suppose that $d(v) = 4$. If $wx \notin E(G)$ for some $w, x \in E(G)$ and G_v^{wx} satisfies the hypothesis of the theorem then we are done by induction and Lemma 2.4. Thus we may suppose that for each pair w, x of neighbours of v, there exists a subset X of $V - v$ such that $i(X) \geq (5|X| - 8)/2$ and $w, x \in X$. We shall say that such a set is a *critical set covering* w, x.

Claim 5.2. *Suppose $i(X) \geq (5|X| - 9)/2$ for some $X \subseteq V - v$. Then $N(v) \not\subseteq X$.*

Proof: If $N(v) \subseteq X$ then

$$i(X + v) = i(X) + 4 \geq (5|X| - 1)/2 = (5|X + v| - 6)/2.$$

This contradicts (3). •

Let $X_1, X_2, \ldots, X_p$ be a family of maximal critical sets which cover each pair of neighbours of v and such that p is as small as possible. Claim 5.2 implies that $p \geq 2$.

Claim 5.3. $|X_i \cap N(v)| = 2$ *for all $1 \leq i \leq p$, and hence $p = 6$.*

Proof: By Claim 5.2 and the definition of the sets X_i, we have $2 \leq |X_i \cap N(v)| \leq 3$ for all $2 \leq i \leq p$. For a contradiction suppose, without loss of generality, that $|X_1 \cap N(v)| = 3$.

We first show that $|X_1 \cap X_i| = 1$ for all $2 \leq i \leq p$. If this is not the case then, relabelling if necessary, we have $|X_1 \cap X_2| \geq 2$. Then Lemma 2.6 implies that

$$\begin{aligned} i(X_1 \cup X_2) &\geq i(X_1) + i(X_2) - i(X_1 \cap X_2) \\ &\geq (5|X_1| - 8)/2 + (5|X_2| - 8)/2 - (5|X_1 \cap X_2| - 7)/2 \\ &= (5|X_1 \cup X_2| - 9)/2. \end{aligned}$$

This contradicts Claim 5.2 since the definition of the sets X_1, X_2 and the fact that $|X_1 \cap N(v)| \geq 3$ implies that $N(v) \subseteq X_1 \cup X_2$. Hence $|X_1 \cap X_i| = 1$ for all $2 \leq i \leq p$. In particular, $|X_i \cap N(v)| = 2$ for all $2 \leq i \leq p$ and hence $p = 4$. We also have $X_1 \cap X_2 \cap X_3 = \emptyset$.

Thus, by Lemma 2.7, we get

$$i(X_1 \cup X_2 \cup X_3) \geq i(X_1) + i(X_2) + i(X_3) - i(X_2 \cap X_3). \tag{4}$$

Using (3) and the fact that $|X_2 \cap X_3| \geq 1$, we have $i(X_2 \cap X_3) \leq (5|X_2 \cap X_3| - 5)/2$. Since X_1, X_2, X_3 are critical, (4) gives $i(X_1 \cup X_2 \cup X_3) \geq (5|X_1| - 8)/2 + (5|X_2| - 8)/2 + (5|X_3| - 8)/2 - (5|X_2 \cap X_3|)/2 = (5|X_1 \cup X_2 \cup X_3| - 24 + 5 + 10)/2 = (5|X_1 \cup X_2 \cup X_3| - 9)/2$. This contradicts Claim 5.2 since $N(v) \subseteq X_1 \cup X_2 \cup X_3$. •

Claim 5.4. $|X_i \cap X_j| \leq 1$ *for all* $1 \leq i < j \leq 6$.

Proof: By symmetry it is sufficient to consider the pair X_1, X_2. For a contradiction suppose that $|X_1 \cap X_2| \geq 2$.

We first consider the case when $X_1 \cap X_2 \cap N(v) = \emptyset$. By Lemma 2.6 we have

$$i(X_1 \cup X_2) \geq i(X_1) + i(X_2) - i(X_1 \cap X_2).$$

Now (3) and the facts that $|X_1 \cap X_2| \geq 2$ and X_1, X_2 are critical, imply that $i(X_1 \cup X_2) \geq (5|X_1 \cup X_2| - 9)/2$. Since $X_1 \cap X_2 \cap N(v) = \emptyset$, we have $N(v) \subseteq X_1 \cup X_2$. This contradicts Claim 5.2.

Thus $X_1 \cap X_2 \cap N(v) \neq \emptyset$. By Lemma 2.6, $i(X_1 \cup X_2) \geq i(X_1) + i(X_2) - i(X_1 \cap X_2)$. By (3), $i(X_1 \cap X_2) \leq (5|X_1 \cap X_2| - 7)/2$. The fact that X_1 is a maximal critical set now implies that

$$i(X_1 \cup X_2) = (5|X_1 \cap X_2| - 9)/2 \quad (5)$$

$$i(X_1 \cap X_2) = (5|X_1 \cap X_2| - 7)/2. \quad (6)$$

Let $X_1 \cap N(v) = \{w, x\}$ and $X_2 \cap N(v) = \{x, y\}$. Relabelling if necessary we may suppose that $X_3 \cap N(v) = \{w, y\}$. By Lemma 2.6, $i(X_1 \cap X_2) + i(X_3) \leq i((X_1 \cap X_2) \cup X_3) + i(X_1 \cap X_2 \cap X_3)$. Using (6), and the facts that $x \notin X_3$ and X_3 is a maximal critical set, we deduce that $i(X_1 \cap X_2 \cap X_3) \geq (5|X_1 \cap X_2 \cap X_3| - 6)/2$. Hypothesis (3) now implies that $|X_1 \cap X_2 \cap X_3| \leq 1$ and hence $i(X_1 \cap X_2 \cap X_3) = 0$. By Lemma 2.7 we have

$$i(X_1 \cup X_2 \cup X_3) \geq i(X_1) + i(X_2) + i(X_3) - i(X_1 \cap X_2) - i(X_1 \cap X_3) - i(X_2 \cap X_3). \quad (7)$$

Suppose $|X_1 \cap X_2 \cap X_3| = 1$. Then $|X_i \cap X_j| \geq 2$ for all $1 \leq i < j \leq 3$ and by (3), $i(X_i \cap X_j) \leq (5|X_1 \cap X_2| - 7)/2$ for all $1 \leq i < j \leq 3$. Substitution into (7) now gives $i(X_1 \cup X_2 \cup X_3) \geq (5|X_1 \cup X_2 \cup X_3| - 8)/2$. This contradicts the fact that X_1 is a maximal critical set.

Thus $X_1 \cap X_2 \cap X_3 = \emptyset$. By (3) and the facts that $|X_1 \cap X_3| \geq 1$ and $|X_2 \cap X_3| \geq 1$, we have

$$i(X_1 \cap X_3) \leq (5|X_1 \cap X_2| - 5)/2, \quad (8)$$

$$i(X_2 \cap X_3) \leq (5|X_1 \cap X_2| - 5)/2. \quad (9)$$

Substituting (6), (8) and (9) into (7) and using the criticality of X_1, X_2, X_3, we obtain $i(X_1 \cup X_2 \cup X_3) \geq (5|X_1 \cup X_2 \cup X_3| - 7)/2$. This again contradicts the fact that X_1 is a maximal critical set and completes the proof of the claim. •

We can now complete the proof of the Theorem. By Claims 5.3 and 5.4,

$$i(\cup_{i=1}^6 X_i) \geq \sum_{i=1}^{6} i(X_i),$$

and

$$|\cup_{i=1}^6 X_i| \leq (\sum_{i=1}^{6} |X_i|) - 8.$$

Since each set X_i is critical we have

$$i(\cup_{i=1}^6 X_i) \geq \sum_{i=1}^{6} i(X_i) = \sum_{i=1}^{6} (5|X_i| - 8)/2 \geq (5|\cup_{i=1}^6 X_i| - 8)/2.$$

This contradicts the fact that X_1 is a maximal critical set. •

We conjecture that the multiplicative constant in the upper bound on $i(X)$ can be weakened in the previous theorem as follows. It is well-known that there exist M-dependent graphs $G = (V, E)$ with $i(X) \leq 3|X| - 6$ for all $X \subseteq V$, $|X| \geq 3$. We also have M-dependent examples for $i(X) \leq 3|X| - 7$, but perhaps graphs satisfying $i(X) \leq 3|X| - 8$ for all $X \subseteq V$, $|X| \geq 5$ are M-independent.

6 Highly connected graphs

By using the proof method of Lovász and Yemini [4, Theorem 2], we shall prove that every $(4k-2)$-connected graph G has rank $k|V(G)| - (2k-1)$ in $\mathcal{N}_k(G)$. We shall use the following elementary lemma on integers.

Lemma 6.1. *Suppose k is a positive integer and let $a_1, a_2, \ldots, a_t$ be integers such that $t \geq 2$, $a_i \geq 2$ for $1 \leq i \leq t$, and $\sum_{i=1}^t (a_i - 1) \geq 4k - 2$. Then*

$$g(a_1, a_2, \ldots, a_t) := \sum_{i=1}^{t} \left(k - \frac{2k-1}{a_i} \right) \geq k.$$

Proof: We may suppose that $a_1, a_2, \ldots, a_t$ have been chosen such that $g(a_1, a_2, \ldots, a_t)$ is as small as possible. Relabelling if necessary we have $a_1 \geq a_i$ for all $2 \leq i \leq t$. If $t \geq 2k$, then the fact that $a_i \geq 2$ for all $1 \leq i \leq t$ implies that $g(a_1, a_2, \ldots, a_t) \geq k$. Hence we may assume that $t \leq 2k - 1$.

Suppose that $a_j \geq 3$ for some $2 \leq j \leq t$. Relabelling if necessary we may assume that $j = 2$. Let $a'_1 = a_1 + 1$, $a'_2 = a_2 - 1$, and $a'_i = a_i$ for $3 \leq j \leq t$. Since $a_1 \geq a_2$, we have

$$(2k-1)^{-1}[g(a_1, a_2, \ldots, a_t) - g(a'_1, a'_2, \ldots, a'_t)] = \frac{1}{(a_2 - 1)a_2} - \frac{1}{a_1(a_1 + 1)} > 0.$$

This contradicts the minimality of $g(a_1, a_2, \ldots, a_t)$ and hence $a_j = 2$ for all $2 \leq j \leq t$. Since $\sum_{i=1}^t (a_i - 1) \geq 4k - 2$, we have $a_1 \geq 4k - t$.

Suppose $t \geq 3$. Let $a''_1 = a_1 + 1$ and $a''_i = a_i$ for $2 \leq j \leq t-1$. Since $t \leq 2k-1$ we have $a_1 \geq 4k - t \geq 2k+1$. Thus

$$g(a_1, a_2, \ldots, a_t) - g(a''_1, a''_2, \ldots, a''_{t-1}) = -\frac{2k-1}{a_1(a_1+1)} + \frac{1}{2} > 0.$$

This contradicts the minimality of $g(a_1, a_2, \ldots, a_t)$ and hence $t = 2$. We now have $a_1 \geq 4k - t = 4k-2$, $a_2 = 2$ and $g(a_1, a_2) \geq k$. •

Theorem 6.2. *Every $(4k-2)$-connected graph $G = (V, E)$ has $\bar{r}_k(G) = k|V| - (2k-1)$.*

Proof: For a contradiction suppose that the theorem is false and let $G = (V, E)$ be a counterexample (that is, a $(4k-2)$-connected graph with $\bar{r}_k(G) < k|V| - (2k-1)$) with the smallest number of vertices, and subject to this, with the largest number of edges.

By (2), there is a family of subsets $\{X_1, X_2, \ldots, X_t\}$ of V such that $\{E(X_1), E(X_2), \ldots, E(X_t)\}$ partitions E, $|X_i| \geq 2$ for $1 \leq i \leq t$, and

$$\sum_{i=1}^{t}(k|X_i| - (2k-1)) < k|V| - (2k-1). \tag{10}$$

By the maximality of $|E|$, $G[X_i]$ is complete for all $1 \leq i \leq t$.

Claim 6.3. *Each vertex $v \in V$ is contained in at least two sets X_i.*

Proof: Suppose the claim is false. Then, after relabelling if necessary, there exists a vertex v for which $v \in X_t$ and $v \notin X_i$, $1 \leq i \leq t-1$ hold. Then all the edges incident to v are in $G[X_t]$ (hence $|X_t| \geq 4k-1$, since $d(v) \geq 4k-2$) and the neighbours of v induce a complete subgraph of G. Let $G' = G - v$, $X'_i = X_i$, for $1 \leq i \leq t-1$, and $X'_t = X_t - \{v\}$. By (10) we have

$$\begin{aligned}\sum_{i=1}^{t}(k|X'_i| - (2k-1)) &= \sum_{i=1}^{t}(k|X_i| - (2k-1)) - k \\ &< k|V| - (2k-1) - k = k|V(G')| - (2k-1).\end{aligned}$$

Since G is a counterexample with as few vertices as possible, G' cannot be $(4k-2)$-connected. This implies that either G is a complete graph on $4k-1$ vertices or there is a vertex separator of size $4k-2$ in G which contains v. In the former case $X_t = V$ must hold, which contradicts (10). In the latter case, since G is $(4k-2)$-connected, each component of $G - T$ contains at least one neighbour of v. But this is impossible, since the neighbours of v induce a complete graph in G. •

Since G is $(4k-2)$-connected, we have $d(v) \geq 4k-2$ for each $v \in V$, and hence

$$\sum_{X_i \ni v}(|X_i| - 1|) \geq 4k-2. \tag{11}$$

Claim 6.4. *For each vertex $v \in V$ we have*

$$\sum_{X_i \ni v} \left(k - \frac{2k-1}{|X_i|} \right) \geq k. \tag{12}$$

Proof: Without loss of generality suppose that $v \in X_1, ..., X_r$, $v \notin X_{r+1}, ..., X_t$, and $|X_1| \geq |X_2| \geq ... \geq |X_r|$. By Claim 6.3 we have $r \geq 2$. Now the claim follows from (11) and Lemma 6.1 by letting $a_i = |X_i|$. •

To finish the proof we take the sum of (12) over all vertices of G and obtain

$$k|V| \leq \sum_{v \in V} \sum_{X_i \ni v} \left(k - \frac{2k-1}{|X_i|} \right) = \sum_{i=1}^{t} |X_i| \left(k - \frac{2k-1}{|X_i|} \right) = \sum_{i=1}^{t} (k|X_i| - (2k-1)).$$

This contradicts (10) and completes the proof. •

Using Theorem 6.2 and Corollary 4.3 we obtain:

Corollary 6.5. *Let d be an even integer. Then every $(2d+2)$-connected graph G has $r_d(G) \geq (\frac{d}{2}+1)|V| - (d+1)$.*

The special case $d = 2$ of the corollary (which implies that 6-connected graphs are rigid in two dimensions) was proved by Lovász and Yemini [4, Theorem 2].

7 Pinning down graphs in 3-space

A *pinning set* for a d-dimensional framework (G,p), is a set $P \subseteq V$ such that the $d|V-P|$ columns of $R_d(G,p)$ indexed by $V-P$ are linearly independent. (This definition is motivated by the fact that if P is a pinning set for (G,P), then every smooth deformation of (G,p) which preserves all edge lengths of G and leaves the points $p(u)$, $u \in P$, fixed must leave all points $p(v)$, $v \in V$, fixed. Thus (G,p) becomes 'rigid' when we 'pin down' the vertices in P, see [6, Statement 8.2.1].) The *pinning number*, $pin_d(G,p)$, of (G,p) is defined to be the size of a smallest pinning set for (G,p). Since the pinning number of any two generic frameworks on G is the same, we may define the pinning number of G, $pin_d(G)$, as the *pinning number* of (G,p) of any generic framework (G,p). It is easy to see that $pin_d(G) \leq pin_d(G,p)$ for all frameworks (G,p).

Lovász [3] gives a polynomial time algorithm for computing $pin_2(G,p)$ for any 2-dimensional framework (G,p). This gives rise to a randomised algorithm for computing $pin_2(G)$ in polynomial time, but there is, as yet, no deterministic polynomial algorithm for determining $pin_2(G)$. Mansfield [5] proved that the problem of computing $pin_3(G,p)$ is NP-hard for 3-dimensional frameworks (G,p). He also showed that computing $pin_3(G)$ for some graph G is NP-hard.

By using Theorem 4.2 we shall prove that highly connected graphs have relatively small pinning number in 3-space. This connection between high connectivity and rigidity in 3-space may be considered as a first step towards the Lovász-Yemini conjecture [4], which asserts that sufficiently highly connected (perhaps 12-connected) graphs are rigid in 3-space (and hence their pinning number is three). Note that there exists a family of 11-connected graphs whose pinning number grows linearly with $|V|$.

Theorem 7.1. *Let $G = (V, E)$ be a 10-connected graph. Then $pin_3(G) \leq \frac{3|V|}{4} + 4$.*

Proof: Let (G, p) be a 4-dimensional generic framework on G and $R_4(G, p)$ be the rigidity matrix of (G, p). Since the framework is generic, $rank R_4(G, p) = r_4(G)$. Let C be the set of columns of $R_4(G, p)$ and C_i be the columns in C corresponding to the i'th coordinate, for $1 \leq i \leq 4$.

By Corollary 6.5 it follows that $rank R_4(G, p) \geq 3|V| - 5$. Let I be a set of $3|V| - 5$ linearly independent columns in $R_4(G, p)$. Relabelling if necessary we may suppose that $|I \cap C_4| \leq |I|/4$. Let (G, p') be the projection of (G, p) onto the subspace of $\mathbb{R}^4$ represented by the first three coordinates. Then $R_3(G, p')$ is obtained from $R_4(G, p)$ by deleting the columns of C_4. Thus $R_3(G, p')$ contains at least $3|I|/4 = (9|V| - 15)/4$ columns of I. This implies that I covers at least $|V|/4 - 4$ triples of columns of $R_3(G, p')$ corresponding to some set $Y \subset V$. Hence $V - Y$ is a pinning set for (G, p') and $pin_3(G) \leq pin_3(G, p') \leq \frac{3|V|}{4} + 4$. •

References

[1] J. Graver, B. Servatius, and H. Servatius, Combinatorial Rigidity, AMS Graduate Studies in Mathematics Vol. 2, 1993.

[2] G. Laman, On graphs and rigidity of plane skeletal structures, *J. Engineering Math.* 4 (1970), 331-340.

[3] L. Lovász, Matroid matching and some applications, *J. Combin. Theory Ser. B* 28 (1980), no. 2, 208-236.

[4] L. Lovász and Y. Yemini, On generic rigidity in the plane, *SIAM J. Algebraic Discrete Methods* 3 (1982), no. 1, 91–98.

[5] A. Mansfield, On the computational complexity of a rigidity problem, *IMA J. Appl. Math.* 27 (1981), no. 4, 423-429.

[6] A. Recski, *Matroid theory and its applications in electric network theory and in statics*, Akadémiai Kiadó, Budapest, 1989.

[7] W. Whiteley, Some matroids from discrete applied geometry, in Matroid theory (J.E. Bonin, J.G. Oxley and B. Servatius eds., Seattle, WA, 1995), Contemp. Math., 197, Amer. Math. Soc., Providence, RI, 1996, 171–311.