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# Connected rigidity matroids and unique realizations of graphs

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Bill Jackson* and Tibor Jordán**

**Abstract**

A $d$-dimensional *framework* is a straight line embedding of a graph $G$ in $\mathbb{R}^d$. We shall only consider *generic* frameworks, in which the co-ordinates of all the vertices of $G$ are algebraically independent. Two frameworks for $G$ are *equivalent* if corresponding edges in the two frameworks have the same length. A framework is a *unique realization* of $G$ in $\mathbb{R}^d$ if every equivalent framework can be obtained from it by a rigid congruence of $\mathbb{R}^d$. Bruce Hendrickson proved that if $G$ has a unique realization in $\mathbb{R}^d$ then $G$ is $(d+1)$-connected and redundantly rigid. He conjectured that every realization of a $(d+1)$-connected and redundantly rigid graph in $\mathbb{R}^d$ is unique. This conjecture is true for $d = 1$ but was disproved by Robert Connelly for $d \geq 3$. We resolve the remaining open case by showing that Hendrickson's conjecture is true for $d = 2$. As a corollary we deduce that every realization of a 6-connected graph as a 2-dimensional generic framework is a unique realization. Our proof is based on a new inductive characterization of 3-connected graphs whose rigidity matroid is connected.

## 1 Introduction

We shall consider finite graphs without loops, multiple edges or isolated vertices. A $d$-dimensional *framework* is a pair $(G, p)$, where $G = (V, E)$ is a graph and $p$ is a map from $V$ to $\mathbb{R}^d$. We consider the framework to be a straight line embedding of $G$ in $\mathbb{R}^d$ in which the *length* of an edge $uv \in E$ is given by the Euclidean distance between the points $p(u)$ and $p(v)$. Two frameworks $(G, p)$ and $(G, q)$ are *equivalent* if corresponding edges of the two frameworks have the same length. We say that two frameworks $(G, p)$, $(G, q)$ are *congruent* if there is a rigid congruence (i.e. translation or rotation) of $\mathbb{R}^d$ which maps $p(v)$ onto $q(v)$ for each $v \in V$. We shall say that $(G, p)$ is a *unique realization* of $G$ in $\mathbb{R}^d$ if every framework which is equivalent to $(G, p)$ is congruent to $(G, p)$. The *unique realization problem* is to decide whether a given realization

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is unique. Saxe [14] proved that this problem is NP-hard. We obtain a problem of different type, however, if we exclude 'degenerate' cases. A framework $(G, p)$ is said to be *generic* if the coordinates of all the points are algebraically independent over the rationals. In what follows we shall consider the unique realization problem for generic frameworks.

A simple necessary condition for unique realization of generic frameworks is rigidity. Intuitively, this means that if we think of a $d$-dimensional framework $(G, p)$ as a collection of bars and joints where points correspond to joints and each edge to a rigid bar joining its end-points, then the framework is rigid if it has no non-trivial continuous deformations. It is known [18] that rigidity is a generic property, that is, the rigidity of $(G, p)$ depends only on the graph $G$, if $(G, p)$ is generic. We say that the graph $G$ is rigid in $\mathbb{R}^d$ if every generic realization of $G$ in $\mathbb{R}^d$ is rigid. (A combinatorial definition for the rigidity of $G$ in $\mathbb{R}^2$ will be given in Section 2 of this paper. We refer the reader to [18, 19] for a formal definition and detailed survey of the rigidity of $d$-dimensional frameworks.)

The necessary condition of rigidity was strengthened by Hendrickson [9] as follows. A graph $G$ is 2-rigid in $\mathbb{R}^d$ if deleting any edge of $G$ results in a graph which is rigid in $\mathbb{R}^d$. (Other authors have used the terms redundantly rigid and edge birigid for 2-rigid.) By using methods from differential topology, Hendrickson proved that the 2-rigidity of $G$ is a stronger necessary condition for the unique realizability of a generic framework $(G, p)$.

Hendrickson [9] also pointed out that the $(d + 1)$-connectivity of $G$ is another necessary condition for a $d$-dimensional framework $(G, p)$ to be a unique realization of $G$: if $G$ has at least $d + 2$ vertices and has a vertex separator of size $d$, then we can obtain an equivalent framework to $(G, p)$ by reflecting $G$ along this separator. Summarising we have

**Theorem 1.1.** *[9] If a generic framework $(G, p)$ is a unique realization of $G$ in $\mathbb{R}^d$ then either $G$ is the complete graph with at most $d + 1$ vertices, or the following conditions hold:*
*(a) $G$ is $(d + 1)$–connected, and*
*(b) $G$ is 2-rigid.*

Hendrickson [7, 8, 9] conjectured that conditions (a) and (b) are sufficient to guarantee that any generic framework $(G, p)$ is a unique realization of $G$. This conjecture is easy to prove for $d = 1$ since $G$ is rigid in $\mathbb{R}$ if and only if $G$ is connected; $G$ is 2-rigid in $\mathbb{R}$ if and only if $G$ is 2-edge-connected; and $(G, p)$ is a unique generic realization of $G$ in $\mathbb{R}$ if and only if $G$ is 2-connected. On the other hand, Connelly [3] has shown that Hendrickson's conjecture is false for $d \geq 3$. We shall settle the remaining case by showing that the conjecture is true for $d = 2$. As a corollary we deduce that unique realizability is also a generic property, that is to say the unique realizability of a 2-dimensional generic framework $(G, p)$ depends only on the graph $G$. Following Connelly [3], we say that a graph $G$ is *globally rigid* in $\mathbb{R}^d$ if every generic realization of $G$ in $\mathbb{R}^d$ is a unique realization. Our solution of the conjecture implies that $G$ is globally rigid in $\mathbb{R}^2$ if and only if $G$ is a complete graph on at most three vertices or $G$ is 3-connected and 2-rigid.

Our proof of the conjecture is based on an inductive construction for all 3-connected 2-rigid graphs. We shall show that every graph in this family can be built up from $K_4$ (which is globally rigid) by an appropriate sequence of operations, where each of the two operations we use preserves global rigidity.

One operation is *edge addition*: we add a new edge connecting some pair of non-adjacent vertices. The other is *1-extension*: we subdivide an edge $uv$ by a new vertex $z$, and add a new edge $zw$ for some $w \neq u, v$. Clearly, the first operation preserves global rigidity. So does the second. This fact follows from a deep result of Connelly [4] (see also [2],[8]), who developed a sufficient condition for the global rigidity of a generic framework in terms of the rank of its 'stress matrix'. Based on this condition, he proved that if $G$ is globally rigid in $\mathbb{R}^2$ and $G'$ is obtained from $G$ by a 1-extension, then $G'$ is also globally rigid in $\mathbb{R}^2$.

In what follows we shall assume that $d = 2$. In this case both conditions in Hendrickson's conjecture can be characterized (and efficiently tested) by purely combinatorial methods. This is straightforward for 3-connectivity. In the case of 2-rigidity, the combinatorial characterization and algorithm are based on the following result of Laman [11]. For a graph $(G, E)$ and a subset $X \subseteq V$ let $i_G(X)$ (or simply $i(X)$ when it is obvious to which graph we are referring) denote the number of edges in the subgraph induced by $X$ in $G$. The graph $G$ is said to be minimally rigid if $G$ is rigid, and $G - e$ is not rigid for all $e \in E$.

**Theorem 1.2.** *[11] A graph $G = (V, E)$ is minimally rigid in $\mathbb{R}^2$ if and only if $|E| = 2|V| - 3$ and*

$$i(X) \leq 2|X| - 3 \text{ for all } X \subset V \text{ with } |X| \geq 2. \tag{1}$$

Note that a graph is rigid if and only if it has a minimally rigid spanning subgraph.

It can be seen from Theorem 1.2 that a 2-rigid graph $G = (V, E)$ will have at least four vertices and at least $2|V| - 2$ edges. We call graphs which are 2-rigid and have this minimum number of edges $M$-circuits. Motivated by Hendrickson's conjecture, Connelly conjectured (see e.g. [6, p.99], [18, p.188]) in the 1980's that all 3-connected $M$-circuits can be obtained from $K_4$ by 1-extensions. It is easy to see that the 1-extension operation preserves 3-connectivity and that it creates an $M$-circuit from an $M$-circuit. The other direction is more difficult. It is equivalent to saying that every 3-connected $M$-circuit on at least five vertices has a vertex of degree three which can be "suppressed" by the inverse operation to 1-extension, so that the resulting graph is a smaller 3-connected $M$-circuit.

The inverse operation to 1-extension is called *splitting off*: it chooses a vertex $v$ of degree three in a graph $G$, deletes $v$ (and the edges incident to $v$) and adds a new edge connecting two non-adjacent neighbours of $v$. If $G$ is a 3-connected $M$-circuit with at least five vertices and at least one of the splittings of $v$ results in a 3-connected $M$-circuit, then we say that the vertex $v$ is *feasible*. It can be seen that each $M$-circuit $G$ has at least four vertices of degree three. It is not true, however, that each vertex of degree three in $G$ is feasible. The existence of such a vertex was verified by Berg and the second named author [1] in their recent solution to Connelly's conjecture.

In this paper we shall show that every 3-connected 2-rigid graph can be obtained from $K_4$ by edge additions and 1-extensions by extending the methods in [1]. We show that every 3-connected 2-rigid graph $G$ on at least five vertices either contains an edge $e$ such that $G - e$ is 3-connected and 2-rigid, or a vertex $v$ of degree three such that splitting off $v$ in $G$ results in a graph which is 3-connected and 2-rigid.

The structure of the paper is as follows. In Section 2 we review elementary results on rigidity: we define the rigidity matroid of a graph and use it to give combinatorial definitions for when a graph is rigid, 2-rigid or an $M$-circuit. In Section 3 we characterize $M$-connected graphs (graphs with a connected rigidity matroid). Section 4 describes and extends lemmas from [1] on splitting off in $M$-circuits. In Section 5, we use the concept of an ear decomposition of a matroid to extend the splitting off theorem of [1] from $M$-circuits to $M$-connected graphs. We use this in Section 6 to obtain our above mentioned recursive construction for 3-connected 2-rigid graphs and hence solve Hendrickson's conjecture.

# 2 Rigid graphs and the rigidity matroid

In this section we prove a number of preliminary lemmas and basic results, most of which are known. Our goal is to make the paper self-contained and to give a unified picture of these frequently used statements. Our proofs are based on Laman's theorem and use only graph theoretical arguments. Some of these results can be found in [6, 12, 16, 18, 19].

Let $G = (V, E)$ be a graph. Let $S$ be a non-empty subset of $E$, and $H$ be the subgraph of $G$ induced by edge set $S$. We say that $S$ is *independent* if

$$i_H(X) \leq 2|X| - 3 \text{ for all } X \subseteq V(H) \text{ with } |X| \geq 2. \tag{2}$$

The empty set is also defined to be independent. The *rigidity matroid* $\mathcal{M}(G) = (E, \mathcal{I})$ is defined on the edge set of $G$ by

$$\mathcal{I} = \{S \subseteq E : S \text{ is independent in } G\}.$$

To see that $\mathcal{M}(G)$ is indeed a matroid, we shall verify that the following three matroid axioms are satisfied. (For basic matroid definitions not given here the reader may consult the book [13].)

(M1) $\emptyset \in \mathcal{I}$,

(M2) if $Y \subset X \in \mathcal{I}$ then $Y \in \mathcal{I}$,

(M3) for every $E' \subseteq E$ the maximal independent subsets of $E'$ have the same cardinality.

Let $G = (V, E)$ be a graph. For $X, Y, Z \subset V$, let $E(X)$ be the set of edges of $G[X]$, $d(X, Y) = |E(X \cup Y) - (E(X) \cup E(Y))|$, and $d(X, Y, Z) = |E(X \cup Y \cup Z) - (E(X) \cup E(Y) \cup E(Z))|$. We define the *degree* of $X$ by $d(X) = d(X, V - X)$. The degree of a vertex $v$ is simply denoted by $d(v)$. We shall need the following equalities, which are easy to check by counting the contribution of an edge to each of their two sides.

**Lemma 2.1.** *Let $G$ be a graph and $X, Y \subseteq V(G)$. Then*

$$i(X) + i(Y) + d(X, Y) = i(X \cup Y) + i(X \cap Y). \tag{3}$$

**Lemma 2.2.** *Let $G$ be a graph and $X, Y, Z \subseteq V(G)$. Then*

$$\begin{aligned} i(X) + i(Y) + i(Z) + d(X, Y, Z) \quad = \quad & i(X \cup Y \cup Z) + i(X \cap Y) + i(X \cap Z) + \\ & i(Y \cap Z) - i(X \cap Y \cap Z). \end{aligned}$$

We say that the graph $H = (V, F)$ is *$M$-independent* if $F$ is independent in $\mathcal{M}(H)$. We call a set $X \subseteq V$ *critical* if $i(X) = 2|X| - 3$ holds.

**Lemma 2.3.** *Let $H = (V, F)$ be $M$-independent and let $X, Y \subset V$ be critical sets in $H$ with $|X \cap Y| \geq 2$. Then $X \cap Y$ and $X \cup Y$ are also critical, and $d(X, Y) = 0$.*

**Proof:** Since $H$ is $M$-independent, (2) holds. By (3) we have
$2|X| - 3 + 2|Y| - 3 = i(X) + i(Y) = i(X \cap Y) + i(X \cup Y) - d(X, Y) \leq$
$2|X \cap Y| - 3 + 2|X \cup Y| - 3 - d(X, Y) = 2|X| - 3 + 2|Y| - 3 - d(X, Y)$. Thus $d(X, Y) = 0$ and equality holds everywhere. Therefore $X \cap Y$ and $X \cup Y$ are also critical. •

**Lemma 2.4.** *Let $G = (V, E')$ be a graph with $|E'| \geq 1$ and let $F \subseteq E'$ be a maximal independent subset of $E'$. Then*

$$|F| = \min\{\sum_{i=1}^{t}(2|X_i| - 3)\} \tag{4}$$

*where the minimum is taken over all collections of subsets $\{X_1, X_2, \ldots, X_t\}$ of $V(G)$ such that $\{E(X_1), E(X_2), \ldots, E(X_t)\}$ partitions $E'$.*

**Proof:** Since $F$ is independent, we have $|F \cap E(X_i)| \leq 2|X_i| - 3$ for all $1 \leq i \leq t$. Thus $|F| \leq \sum_{i=1}^{t}(2|X_i| - 3)$ for any collection of subsets $\{X_1, X_2, ..., X_t\}$ satisfying the hypothesis of the lemma.

To see that equality can be attained, let $H$ be the subgraph of $G$ induced by $F$. Consider the maximal critical sets $X_1, X_2, ..., X_t$ in $H$. By Lemma 2.3 we have $|X_i \cap X_j| \leq 1$ for all $1 \leq i < j \leq t$. Since every single edge of $F$ induces a critical set, it follows that $\{E_H(X_1), E_H(X_2), ..., E_H(X_t)\}$ is a partition of $F$. Thus

$$|F| = \sum_{1}^{t} |E_H(X_i)| = \sum_{1}^{t}(2|X_i| - 3).$$

To complete the proof we show that $\{E_G(X_1), E_G(X_2), ..., E_G(X_t)\}$ is a partition of $E'$. Choose $uv \in E' - F$. Since $F$ is a maximal independent subset of $E'$, $F + uv$ is dependent. Thus there exists a set $X \subseteq V$ such that $u, v \in X$ and $i_H(X) = 2|X| - 3$. Hence $X$ is a critical set in $H$. This implies that $X \subseteq X_i$ and hence $uv \in E_G(X_i)$ for some $1 \leq i \leq t$. •

It follows from the definition of independence that $\mathcal{M}(G)$ satisfies axioms (M1) and (M2). Lemma 2.4 implies that $\mathcal{M}(G)$ also satisfies (M3). It also determines the rank function of $\mathcal{M}(G)$, which we shall denote by $r_G$ or simply by $r$.

**Corollary 2.5.** *First Let $G = (V, E)$ be a graph. Then $\mathcal{M}(G)$ is a matroid, in which the rank of a non-empty set $E' \subseteq E$ of edges is given by*

$$r(E') = \min\{\sum_{i=1}^{t}(2|X_i)| - 3)\}$$

*where the minimum is taken over all collections of subsets $\{X_1, X_2, \ldots, X_t\}$ of $G$ such that $\{E(X_1), E(X_2), \ldots, E(X_t)\}$ partitions $E'$.*

We say that a graph $G = (V, E)$ is *rigid* if $r(E) = 2|V| - 3$ in $(\mathcal{M})(G)$. The graph $G$ is *minimally rigid* if it is rigid and $|E| = 2|V| - 3$. Thus, if $G$ is rigid and $H = (V, E')$ is a spanning subgraph of $G$, then $H$ is minimally rigid if and only if $E'$ is a base in $\mathcal{M}(G)$. Theorem 1.2 ensures that these definitions agree with the intuitive definitions for rigidity given in Section 1.

A *$k$-separation* of a graph $H = (V, E)$ is a pair $(H_1, H_2)$ of edge-disjoint subgraphs of $G$ each with at least $k+1$ vertices such that $H = H_1 \cup H_2$ and $|V(H_1) \cap V(H_2)| = k$. The graph $H$ is said to be *$k$-connected* if it has at least $k + 1$ vertices and has no $j$-separation for all $0 \leq j \leq k - 1$. If $(H_1, H_2)$ is a $k$-separation of $H$, then we say that $V(H_1) \cap V(H_2)$ is a *$k$-separator* of $H$. For $X \subseteq V$ let $N(X)$ denote the set of *neighbours* of $X$ (that is, $N(X) := \{v \in V - X : uv \in E \text{ for some } u \in X\}$).

## 2.1 Minimally rigid graphs

We first investigate the connectivity properties of minimally rigid graphs.

**Lemma 2.6.** *Let $G = (V, E)$ be minimally rigid with $|V| \geq 3$. Then*
*(a) $G$ is 2-connected.*
*(b) for every $\emptyset \neq X \subset V$ we have $d(X) \geq 2$ and if $d(X) = 2$ holds then either $|X| = 1$ or $|V - X| = 1$,*

**Proof:** Suppose that for some $v \in V$ the graph $G - v$ is disconnected and let $A \cup B$ be a partition of $V - v$ with $d(A, B) = 0$. Then (2) gives $|E| = 2|V| - 3 = i(A+v) + i(B+v) \leq 2(|A|+1) - 3 + 2(|B|+1) - 3 = 2(|A|+|B|+1) - 4 = 2|V| - 4$, a contradiction. This proves (a).

Using (a), we have $d(X) \geq 2$ for every $\emptyset \neq X \subset V$. Suppose $|X|, |V - X| \geq 2$. By (2) we obtain $|E| = i(X) + i(V - X) + d(X) \leq 2|X| - 3 + 2|V - X| - 3 + d(X) = 2|V| - 6 + d(X) = |E| - 3 + d(X)$. This implies $d(X) \geq 3$ and proves (b). •

Let $v$ be a vertex in a graph $G$ with $d(v) = 3$ and $N(v) = \{u, w, z\}$. Recall that the operation *splitting off* means deleting $v$ (and the edges incident to $v$) and adding a new edge, say $uw$, connecting two non-adjacent vertices of $N(v)$. The resulting graph is denoted by $G_v^{uw}$ and we say that the splitting is made *on the pair* $uv, wv$. Note that $v$ can be split off in at most three different ways.

Let $G = (V, E)$ be minimally rigid and let $v$ be a vertex with $d(v) = 3$. Splitting off $v$ on the pair $uv, wv$ is said to be *suitable* if $G_v^{uw}$ is minimally rigid. We also call a vertex $v$ *suitable* if there is an suitable splitting at $v$. We shall show that every vertex of degree three in a minimally rigid graph is suitable.

**Lemma 2.7.** *Let $G = (V, E)$ be minimally rigid and let $X, Y, Z \subset V$ be critical sets in $G$ with $|X \cap Y| = |X \cap Z| = |Y \cap Z| = 1$ and $X \cap Y \cap Z = \emptyset$. Then $X \cup Y \cup Z$ is critical, and $d(X, Y, Z) = 0$.*

**Proof:** Since $G$ is minimally rigid and our sets are critical, Lemma 2.2 gives $2|X| - 3 + 2|Y| - 3 + 2|Z| - 3 + d(X, Y, Z) = i(X) + i(Y) + i(Z) + d(X, Y, Z) \leq i(X \cup Y \cup Z) \leq 2(|X \cup Y \cup Z|) - 3 = 2(|X| + |Y| + |Z| - 3) - 3 = 2|X| - 3 + 2|Y| - 3 + 2|Z| - 3$. Hence $d(X, Y, Z) = 0$ and equality holds everywhere. Thus $X \cup Y \cup Z$ is critical. •

**Lemma 2.8.** *Let $v$ be a vertex in an minimally rigid graph $G = (V, E)$.*
*(a) If $d(v) = 2$ then $G - v$ is minimally rigid.*
*(b) If $d(v) = 3$ then $v$ is suitable.*

**Proof:** Part (a) follows easily from (2) and from the definition of minimally rigid graphs.

To prove (b) let $N(v) = \{u, w, z\}$. It is easy to see that splitting off $v$ on the pair $uv, wv$ is not suitable if and only if there exists a critical set $X \subset V$ with $u, w \in X$ and $v, z \notin X$. Also observe that no critical set $Z \subseteq V - v$ can satisfy $d(v, Z) \geq 3$, since otherwise $E(G[Z \cup \{v\}])$ is not independent in $G$, contradicting the fact that $G$ is minimally rigid. Thus if $v$ is not suitable then there exist maximal critical sets $X_{uw}, X_{uz}, X_{wz} \subset V - v$ each containing precisely two neighbours ($\{u, w\}, \{u, z\}, \{wz\}$, resp.) of $v$. By Lemma 2.3 and the maximality of these sets we must have $|X_{uw} \cap X_{uz}| = |X_{uw} \cap X_{wz}| = |X_{uz} \cap X_{wz}| = 1$. Thus, by Lemma 2.7 the set $Y := X_{uw} \cup X_{uz} \cup X_{wz}$ is also critical. Since $N(v) \subseteq Y$, we have $d(v, Y) \geq 3$. This is impossible by our previous observation. Therefore $v$ is suitable. •

The minimally rigid graph $K_4 - e$ shows that among the three possible splittings at a vertex of degree three there may be only one which is suitable.

We now define the reverse operations of vertex deletion and vertex splitting used in Lemma 2.8. The operation 0-*extension* adds a new vertex $v$ and two edges $vu, vw$ with $u \neq w$. The operation 1-*extension* subdivides an edge $uw$ by a new vertex $v$ and adds a new edge $vz$ for some $z \neq u, w$. An *extension* is either a 0-extension or a 1-extension. The next lemma follows easily from (2).

**Lemma 2.9.** *Let $G$ be minimally rigid and let $G'$ be obtained from $G$ by an extension. Then $G'$ is minimally rigid.*

**Theorem 2.10.** *Let $G = (V, E)$ be minimally rigid and let $G' = (V', E')$ be a minimally rigid subgraph of $G$. Then $G$ can be obtained from $G'$ by a sequence of extensions.*

**Proof:** We shall prove that $G'$ can be obtained from $G$ by a sequence of vertex splittings and deletions of vertices (of degree two). The theorem will then follow since these are the inverse operations of extensions.

The proof is by induction on $|V - V'|$. Since $G'$ is minimally rigid, it must be an induced subgraph of $G$. Thus the theorem holds trivially when $|V - V'| = 0$. Now suppose that $Y = V - V' \neq \emptyset$. Since $G'$ and $G$ are minimally rigid, it is easy to see that $|E - E'| = 2|Y|$ holds. Therefore, if $|Y| = 1$, then we must have $d(v) = 2$ for the unique vertex $v \in Y$. Hence, by Lemma 2.8(a), $G - v$ is a minimally rigid subgraph of $G$ which contains $G'$ and has $|V(G - v) - V'| < |V - V'|$, and the theorem follows by induction. Thus we may assume that $|Y| \geq 2$.

**Claim 2.11.** *If $|Y| \geq 2$ then $\sum_{v \in Y} d(v) \leq 4|Y| - 3$.*

**Proof:** Since $|V'| \geq 2$ and $|V - V'| \geq 2$, we can apply Lemma 2.6(b) to deduce that $d(Y) \geq 3$. Since $i(Y) + d(Y) = |E - E'| = 2|Y|$, we obtain

$$\sum_{v \in Y} d(v) = 2i(Y) + d(Y) = 4|Y| - d(Y) \leq 4|Y| - 3.$$

•

It follows from Claim 2.11 (and from the fact that the minimum degree in $G$ is at least two) that there is a vertex $v \in Y$ with $2 \leq d(v) \leq 3$. Now Lemma 2.8 implies that either $G - v$ or, for some edges $vu, vw$, $G_v^{uw}$ is a minimally rigid proper subgraph of $G$ which contains $G'$. As above, the theorem follows by induction. •

By choosing $G'$ to be an arbitrary edge of $G$ we obtain the following constructive characterization of minimally rigid graphs (called the Henneberg or Henneberg-Laman construction, c.f. [10, 11]).

**Corollary 2.12.** *$G = (V, E)$ is minimally rigid if and only if $G$ can be obtained from $K_2$ by a sequence of extensions.*

**Theorem 2.13.** *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two minimally rigid graphs with $|V_1 \cap V_2| \geq 2$. Then $G_1 \cup G_2$ is rigid. Moreover, if $G_1 \cap G_2$ is minimally rigid then $G_1 \cup G_2$ is minimally rigid as well.*

**Proof:** Let $F'$ be a maximal independent set in $\mathcal{M}(G_1 \cap G_2)$. Let $K_t$ be the complete graph with vertex set $V(G_1 \cap G_2)$ and $F$ be a basis of $\mathcal{M}(K_t)$ containing $F'$. Let $H$ be a minimally rigid spanning subgraph of $G_2 + (F - F')$ which contains $F$. Such an $H$ exists, since $G_2$, and hence $G_2 + (F - F')$, is rigid. (To see that $F$ and $H$ exist we use the fact that any independent set in a matroid can be extended to a basis.) Now Theorem 2.10 implies that $H$ can be obtained by a sequence of extensions from $(V_1 \cap V_2, F)$. The same sequence of extensions, applied to $G_1$, yields a minimally rigid spanning subgraph of $G_1 \cup G_2$ by Lemma 2.9. This proves that $G_1 \cup G_2$ is rigid.

The second assertion follows from the fact that if $G_1 \cap G_2$ is minimally rigid then $F = F'$ and $H = G_2$. •

**Corollary 2.14.** *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two rigid graphs with $|V_1 \cap V_2| \geq 2$. Then $G_1 \cup G_2$ is rigid.*

Let $G = (V, E)$ be a graph. Since every edge of $G$ induces a rigid subgraph of $G$, Corollary 2.14 implies that the maximal rigid subgraphs $R_1, R_2, ..., R_t$ (called the *rigid components of* $G$) of $G$ are pairwise edge-disjoint and $E(R_1), E(R_2), ..., E(R_t)$ is a partition of $E$. Thus a graph is rigid if and only if it has precisely one rigid component.

## 2.2 $M$-circuits and 2-rigid graphs

Given a graph $G = (V, E)$, a subgraph $H = (W, C)$ is said to be an *$M$-circuit* in $G$ if $C$ is a circuit (i.e. a minimal dependent set) in $\mathcal{M}(G)$. In particular, $G$ is an *$M$-circuit* if $E$ is a circuit in $\mathcal{M}(G)$. For example, $K_4$, $K_{3,3}$ plus an edge, and $K_{3,4}$ are all $M$-circuits. Using (2) we may deduce:

**Lemma 2.15.** *Let $G = (V, E)$ be a graph. The following statements are equivalent.*
*(a) $G$ is an $M$-circuit.*
*(b) $|E| = 2|V| - 2$ and $G - e$ is minimally rigid for all $e \in E$.*
*(c) $|E| = 2|V| - 2$ and $i(X) \leq 2|X| - 3$ for all $X \subseteq V$ with $2 \leq |X| \leq |V| - 1$.*

We shall need the following elementary properties of $M$-circuits which can be derived in a similar way to Lemma 2.6.

**Lemma 2.16.** *[1, Lemma 2.4] Let $H = (V, E)$ be an $M$-circuit.*
*(a) For every $\emptyset \neq X \subset V$ we have $d(X) \geq 3$ and if $d(X) = 3$ holds then either $|X| = 1$ or $|V - X| = 1$,*
*(b) If $X \subset V$ is critical with $|X| \geq 3$ then $H[X]$ is 2-connected.*

Let $H = (V, E)$ be a 2-connected graph and suppose that $(H_1, H_2)$ is a 2-separation of $G$ with $V(H_1) \cap V(H_2) = \{a, b\}$. For $1 \leq i \leq 2$, let $H_i' = H_i + ab$ if $ab \notin E(H_i)$ and otherwise put $H_i' = H_i$. We say that $H_1, H_2$ are the *cleavage graphs* obtained by *cleaving $G$ along* $\{a, b\}$. The inverse operation of cleaving is 2-sum: given two graphs $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ with $V_1 \cap V_2 = \emptyset$ and two designated edges $u_1v_1 \in E_1$ and $u_2v_2 \in E_2$, the 2-*sum* of $H_1$ and $H_2$ (along the edge pair $u_1v_1$, $u_2v_2$), denoted by $H_1 \oplus_2 H_2$, is the graph obtained from $H_1 - u_1v_1$ and $H_2 - u_2v_2$ by identifying $u_1$ with $u_2$ and $v_1$ with $v_2$. We shall use the following results on 2-sums and 2-separations.

**Lemma 2.17.** *[1, Lemma 4.1] Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be $M$-circuits and let $u_1v_1 \in E_1$ and $u_2v_2 \in E_2$. Then the 2-sum $G_1 \oplus_2 G_2$ along the edge pair $u_1v_1$, $u_2v_2$ is an $M$-circuit.*

**Lemma 2.18.** *[1, Lemma 4.2] Let $G = (V, E)$ be an $M$-circuit and let $G'$ and $G''$ be the graphs obtained from $G$ by cleaving $G$ along a 2-separator. Then $G'$ and $G''$ are both $M$-circuits.*

Recall that a graph $G$ is *2-rigid* if $G$ has at least two edges and $G - e$ is rigid for all $e \in E$. $M$-circuits are examples of (minimally) 2-rigid graphs. Note also that a graph $G$ is 2-rigid if and only if $G$ is rigid and each edge of $G$ belongs to a circuit in $\mathcal{M}(G)$ i.e. an $M$-circuit of $G$.

It follows from Theorem 2.13 that any two maximal 2-rigid subgraphs of a graph $G = (V, E)$ can have at most one vertex in common, and hence are edge-disjoint. Defining a *2-rigid component* of $G$ to be either a maximal 2-rigid subgraph of $G$, or a subgraph induced by an edge which belongs to no $M$-circuit of $G$, we deduce that the 2-rigid components of $G$ partition $E$. Since each 2-rigid component is rigid, this partition is a refinement of the partition of $E$ given by the rigid components of $G$.

We shall need two elementary lemmas on 2-rigidity.

**Lemma 2.19.** *If $G$ is 2-rigid and $G'$ is obtained from $G$ by an edge addition or a 1-extension, then $G'$ is 2-rigid.*

**Proof:** This follows from the definition of 2-rigidity and the facts that edge additions, 0-extensions and 1-extensions preserve rigidity. •

**Lemma 2.20.** *If $G$ is 2-rigid and $\{u, v\}$ is a 2-separator in $G$ then $d(u), d(v) \geq 4$.*

**Proof:** Suppose $d(u) \leq 3$. Then we can choose an edge $e$ incident to $u$ such that $G - e$ is not 2-connected. By Lemma 2.6(a), $G - e$ is not rigid. This contradicts the 2-rigidity of $G$. •

# 3 Graphs with a connected rigidity matroid

Given a matroid $\mathcal{M} = (E, \mathcal{I})$, we define a relation on $E$ by saying that $e, f \in E$ are related if $e = f$ or if there is a circuit $C$ in $M$ with $e, f \in C$. It is well-known that this is an equivalence relation. The equivalence classes are called the *components* of $\mathcal{M}$. If $\mathcal{M}$ has at least two elements and only one component then $\mathcal{M}$ is said to be *connected*. If $\mathcal{M}$ has components $E_1, E_2, \ldots, E_t$ and $\mathcal{M}_i$ is the matroid restriction of $\mathcal{M}$ onto $E_i$ then $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2 \ldots \oplus \mathcal{M}_t$ is the direct sum of the $\mathcal{M}_i$'s.

We say that a graph $G = (V, E)$ is *$M$-connected* if $\mathcal{M}(G)$ is connected. For example, $K_{3,m}$ is $M$-connected for all $m \geq 4$. The *$M$-components* of $G$ are the subgraphs of $G$ induced by the components of $\mathcal{M}(G)$. Since the $M$-circuits of $G$ are 2-rigid, every $M$-circuit of $G$ is contained in one of the 2-rigid components of $G$. Thus the partition of $E(G)$ given by the $M$-components is a refinement of the partition given by the 2-rigid components and hence a further refinement of the partition given by the rigid components. Furthermore, $\mathcal{M}(G)$ can be expressed as the direct sum of the rigidity matroids of the rigid components of $G$, the 2-rigid components of $G$, or the $M$-components of $G$.

**Lemma 3.1.** *Suppose that $G$ is $M$-connected. Then $G$ is 2-rigid.*

**Proof:** $G$ is rigid, since otherwise $G$ has at least two rigid components and hence at least two $M$-components. Since $\mathcal{M}(G)$ is connected, every edge $e$ is contained in a circuit of $\mathcal{M}(G)$. Thus $G$ is 2-rigid. •

The main result of this section characterizes which 2-rigid graphs are $M$-connected. We say that a graph $G$ is *nearly 3-connected* if $G$ can be made 3-connected by adding at most one new edge.

**Theorem 3.2.** *Suppose that $G$ is nearly 3-connected and 2-rigid. Then $G$ is $M$-connected.*

**Proof:** For a contradiction suppose that $G$ is not $M$-connected and let $H_1$, $H_2$,...,$H_q$ be the $M$-components of $G$. Let $S_i = V(H_i) - \cup_{j\neq i} V(H_j)$ denote the set of vertices belonging to no other $M$-component than $H_i$, and let $P_i = V(H_i) - S_i$ for $1 \leq i \leq q$. Let $n_i = |V(H_i)|$, $s_i = |S_i|$, $p_i = |P_i|$. Clearly, $n_i = s_i + p_i$ and $|V| = \sum_{i=1}^q s_i + |\cup_{i=1}^q P_i|$. Moreover, we have $\sum_{i=1}^q p_i \geq 2|\cup_{i=1}^q P_i|$. Since every edge of $G$ is in some $M$-circuit, and every $M$-circuit has at least four vertices, we have that $n_i \geq 4$ for $1 \leq i \leq q$. Furthermore, since $G$ is nearly 3-connected, $p_i \geq 2$ for all $1 \leq i \leq q$, and $p_i \geq 3$ for all but at most two $M$-components.

Let us choose a basis $B_i$ in each rigidity matroid $\mathcal{M}(H_i)$. Using the above inequalities we have

$$\begin{aligned} |\cup_{i=1}^q B_i| &= \sum_{i=1}^q |B_i| = \sum_{i=1}^q (2n_i - 3) = 2\sum_{i=1}^q n_i - 3q \geq \\ &\quad (2\sum_{i=1}^q s_i + \sum_{i=1}^q p_i) + \sum_{i=1}^q p_i - 3q \geq 2|V| + 3q - 2 - 3q = 2|V| - 2. \end{aligned}$$

Since $r(\mathcal{M}(G)) = 2|V| - 3$, this implies that $\cup_{i=1}^q B_i$ contains a circuit, contradicting the fact that the $B_i$'s are bases for the $\mathcal{M}(H_i)$'s and $\mathcal{M}(G) = \oplus_{i=1}^q \mathcal{M}(H_i)$. •

A graph $G$ is *birigid* if $G - v$ is rigid for all $v \in V(G)$. It was shown by Servatius [15, Theorem 2.2] (using a similar argument to our proof of Theorem 3.2) that every birigid graph is $M$-connected. Theorem 3.2 extends this result, since birigid graphs are clearly 3-connected and 2-rigid. The wheels (on at least 5 vertices) are 3-connected 2-rigid graphs which are not birigid. This shows that the extension is proper.

We need the following results to complete our characterization of $M$-connected graphs. The first two lemmas follow from Lemmas 2.17 and 2.18, respectively.

**Lemma 3.3.** *Suppose $G_1$ and $G_2$ are $M$-connected. Then $G_1 \oplus_2 G_2$ is $M$-connected.*

**Lemma 3.4.** *Suppose $G_1$ and $G_2$ are obtained from $G$ by cleaving $G$ along a 2-separator. If $G$ is $M$-connected then $G_1$ and $G_2$ are also $M$-connected.*

Let $G = (V, E)$ be a 2-connected graph, $c \geq 3$ be an integer, and let $(X_1, X_2, ..., X_c)$ be cyclically ordered subsets of $V$ satisfying (by taking $X_{c+1} = X_1$):
(i) $|X_i \cap X_j| = 1$, for $|i - j| = 1$, and $X_i \cap X_j = \emptyset$ for $|i - j| \geq 2$, and

(ii) $\{E(X_1), E(X_2), ..., E(X_c)\}$ is a partition of $E$.
Then we say that $(X_1, X_2, ..., X_c)$ is a *polygon (of size c)* in $G$. It is easy to see that if $u$ and $v$ are distinct vertices with $\{u\} = X_{i-1} \cap X_i$ and $\{v\} = X_j \cap X_{j+1}$, for some $1 \leq i, j \leq c$, then either $\{u, v\}$ is a 2-separator in $G$ or $i = j$ and $X_i = \{u, v\}$.

**Lemma 3.5.** *Suppose that $G = (V, E)$ has a polygon of size $c$. Then*
*(a) $G$ is not $M$-connected.*
*(b) If $c \geq 4$ then $G$ is not rigid.*

**Proof:** Let $X_1, X_2, ..., X_c$ be a polygon and let $E_i = E(X_i)$ for $1 \leq i \leq c$. Note that $E_1, E_2, ..., E_c$ is a partition of $E$. Using the polygon structure we obtain

$$r(E) \leq \sum_{i=1}^{c} r(E_i) \leq \sum_{i=1}^{c} (2|X_i| - 3) = 2|V| + 2c - 3c = 2|V| - c. \tag{5}$$

Thus for $c \geq 4$ we have $r(E) \leq 2|V| - 4$, and hence $G$ is not rigid. This proves (b). To prove (a) suppose that $G$ is $M$-connected. Then $G$ is rigid and $r(E) = 2|V| - 3$. By (b) this yields $c = 3$. Moreover, equality must hold everywhere in (5). Thus $r(E) = \sum_{i=1}^{c} r(E_i)$. Thus $\mathcal{M}(\mathcal{G})$ is the direct sum of its restrictions to the sets $E_i$. This contradicts the fact that $\mathcal{M}(G)$ is a connected matroid. •

We say that a 2-separator $\{x_1, x_2\}$ *crosses* another 2-separator $\{y_1, y_2\}$ in a graph $G$, if $x_1$ and $x_2$ are in different components of $G - \{y_1, y_2\}$. It is easy to see that if $\{x_1, x_2\}$ crosses $\{y_1, y_2\}$ then $\{y_1, y_2\}$ crosses $\{x_1, x_2\}$. Thus, we can say that these 2-separators are *crossing*. It is also easy to see that crossing 2-separators induce a polygon of size four in $G$. Thus Lemma 3.5(a) has the following corollary:

**Lemma 3.6.** *Suppose that $G$ is rigid (and hence 2-connected). Then there are no crossing 2-separators in $G$.*

Let $G = (V, E)$ be a 2-connected graph with no crossing 2-separators. The *cleavage units* of $G$ are the graphs obtained by recursively cleaving $G$ along each of its 2-separators. Since $G$ has no crossing 2-separators this sequence of operations is uniquely defined and results in a unique set of graphs each of which have no 2-separators. Thus each cleavage unit of $G$ is either 3-connected or else a complete graph on three vertices. The stronger hypothesis that $G$ has no polygons will imply that each cleavage unit of $G$ is a 3-connected graph. In this case, an equivalent definition for the cleavage units is to first construct the *augmented graph* $\hat{G}$ from $G$ by adding all edges $uv$ for which $\{u, v\}$ is a 2-separator of $G$ and $uv \notin E$, and then take the cleavage units to be the maximal 3-connected subgraphs of $\hat{G}$. (These definitions are a special case of a general decomposition theory for 2-connected graphs due to Tutte [17].)

**Theorem 3.7.** *A graph $G$ is $M$-connected if and only if it is 2-connected, has no polygon, and each of its cleavage units is 2-rigid.*

**Proof:** If $G$ is $M$-connected, then $G$ is rigid and hence 2-connected by Lemma 2.6(a), $G$ has no polygons by Lemma 3.6, each cleavage unit of $G$ is $M$-connected by Lemma 3.4, and hence each cleavage unit is 2-rigid by Lemma 3.1. On the other hand, if $G$ is 2-connected, has no polygons and each cleavage unit is 2-rigid, then each cleavage unit is $M$-connected by Theorem 3.2, and $G$ is $M$-connected by Lemma 3.3. •

The weaker hypothesis that $G$ is 2-connected, has no polygons, and is 2-rigid is not sufficient to imply that $G$ is $M$-connected. This can be seen by considering the graph $G$ obtained from the triangular prism $H = K_3 \times K_2$ by replacing each edge $v_iv_j$ of $H$ by a complete graph with vertex set $\{v_i, v_j, v_i', v_j'\}$, where $v_i', v_j' \notin V(H)$. The graph $G$ is 2-rigid since it is rigid and every edge belongs to an $M$-circuit (a complete graph on four vertices). To see that $G$ is not $M$-connected we first note that $H$ is not 2-rigid. This follows since there exists $X \subset V(H)$ with $|X| = 3 = d_H(X)$. Choosing an edge $e$ in $H$ from $X$ to $V(H) - X$, we may use Lemma 2.6(b) to deduce that $H - e$ is not rigid. We may now deduce that $G$ is not $M$-connected since $H$ is a cleavage unit of $G$, and every cleavage unit of an $M$-connected graph is $M$-connected by Lemma 3.4

We close this section by obtaining two further results on $M$-connectivity which we will need later.

**Lemma 3.8.** *Let $G = (V, E)$ be a 2-connected graph and $\{u, v\}$ be a 2-separator of $G$ such that $uv \in E$. Then $G$ is $M$-connected if and only if $H = G - uv$ is $M$-connected.*

**Proof:** This follows from Theorem 3.7 since $G$ is 2-connected if and only if $H$ is 2-connected, $G$ has no polygons if and only if $H$ has no polygons, and $\hat{G} = \hat{H}$ so the cleavage units of $G$ and $H$ are identical. •

**Lemma 3.9.** *If $G$ is $M$-connected and $G'$ is obtained from $G$ by an edge addition or a 1-extension, then $G'$ is $M$-connected.*

**Proof:** Note that since $G$ is $M$-connected, $G$ has no polygons by Lemma 3.6 and each cleavage unit of $G$ is $M$-connected by Lemma 3.4. We proceed by induction on the number of cleavage units of $G$. Suppose $G$ is nearly 3-connected. Then $G'$ is nearly 3-connected. Since $G$ is $M$-connected, $G$ is 2-rigid by Lemma 3.1. Hence $G'$ is 2-rigid by Lemma 2.19. Thus $G'$ is $M$-connected by Theorem 3.2.

Hence we may suppose that $G$ is not nearly 3-connected. Then we can find a 2-separation $(G_1, G_2)$ in $G$ such that both endvertices of $e$ belong to $G_1$ when $G' = G + e$, and such that all neighbours of $v$ belong to $G_1$ when $G'$ is a 1-extension of $G$ by $v$. Let $H_1, H_2$ be the cleavage graphs of $G$ corresponding to $G_1, G_2$, respectively. Then $H_1, H_2$ are $M$-connected by Lemma 3.4. Let $H_1'$ be obtained from $H_1$ in the same way that $G'$ was obtained from $G$. By induction $H_1'$ is $M$-connected. Thus $H' = H_1' \oplus_2 H_2$, is $M$-connected by Lemma 3.3. If $H' = G'$ then we are done. If not, then $G' = H' + xy$, where $\{x, y\} = V(G_1) \cap V(G_2)$, and we are done by Lemma 3.8. •

# 4 Admissible splittings in $M$-circuits

Let $G = (V, E)$ be a graph and let $V_3 = \{v \in V : d(v) = 3\}$. We will refer to vertices in $V_3(G)$ as *nodes* of $G$ and to the subgraph $G[V_3]$ as the *node-subgraph* of $G$. A node of $G$ with degree at most one (exactly two, exactly three) in the node-subgraph of $G$ is called a *leaf node* (*series node*, *branching node*, respectively). A *wheel* $W_n = (V, E)$ is a graph on $n \geq 4$ vertices which has a vertex $z$ which is adjacent to all the other vertices and for which $W_n[V - z]$ is a cycle. Thus the node-subgraph of a wheel $W_n$ with $n \geq 5$ is a cycle. It was shown in [1, Lemma 2.1] that if $G$ is an $M$-circuit then either $G$ is a wheel or $G[V_3]$ is a forest. The proof can be extended to $M$-connected graphs to give:

**Lemma 4.1.** *Let $G$ be $M$-connected. If $G$ is not a wheel, then the nodes of $G$ induce a forest in $G$.*

We also need two results on $M$-circcuits from [1]. The proof of the first lemma is similar to that of Lemma 2.3.

**Lemma 4.2.** *[1, Lemma 2.3] Let $G = (V, E)$ be an $M$-circuit and let $X, Y \subset V$ be critical sets with $|X \cap Y| \geq 2$ and $|X \cup Y| \leq |V| - 1$. Then $X \cap Y$ and $X \cup Y$ are both critical, and $d(X, Y) = 0$.*

**Lemma 4.3.** *[1, Lemma 2.5] Let $H = (V, E)$ be an $M$-circuit and let $X \subset V$ be a critical set. Then $V - X$ contains at least one node. Furthermore, if $|V - X| \geq 2$, then $V - X$ contains at least two nodes.*

We shall say that splitting off a node $v$ in an $M$-connected graph is *admissible* if it preserves $M$-connectivity, that $v$ is an *admissible node* if it has an admissible splitting, and otherwise that $v$ is *non-admissible*. Note that an admissible splitting off in an $M$-circuit results in an $M$-connected graph with $|E| = 2|V| - 2$, and hence results in another $M$-circuit. The following result follows easily from Lemma 2.15.

**Lemma 4.4.** *[1, Lemma 3.1] Let $H = (V, E)$ be an $M$-circuit and $v$ be a node in $G$ with $N(v) = \{u, w, z\}$. Then splitting off $v$ on the pair $uv, wv$ is not admissible if and only if there is a critical set $X \subset V$ with $u, w \in X$ and $v, z \notin X$.*

If $v$ is a node in a graph $G$ with $N(v) = \{u, w, z\}$ and $X$ is a critical set with $u, w \in X$ and $v, z \notin X$ then we call $X$ a *$v$-critical set on $\{u, w\}$*, or simply a *$v$-critical set*. If $X$ is a $v$-critical set on $\{u, w\}$ for some node $v$ with $N(v) = \{u, w, z\}$, and $d(z) \geq 4$, then $X$ is said to be *node-critical*.

Our next lemma extends [1, Lemma 3.2].

**Lemma 4.5.** *Let $H$ be an $M$-circuit, $|V(H)| \geq 5$, and $v$ be a non-admissible leaf node in $H$ with $N(v) = \{x, y, z\}$. Suppose that no two neighbours of $v$ are a 2-separator in $H$.*
*(a) If $z$ is a node of $H$ then for any pair $X_1, X_2$ of $v$-critical sets on $\{y, z\}$, and $\{x, z\}$, respectively, we have $|X_1 \cap X_2| \geq 2$ and $X_1 \cup X_2 = V(H) - v$.*
*(b) If $v$ is not adjacent to a node then there exist two $v$-critical sets $X_1, X_2$ with $|X_1 \cap X_2| \geq 2$, $X_1 \cup X_2 = V(H) - v$.*

**Proof:** We first consider the case when $z$ is a node of $H$. Since $v$ is non-admissible, Lemma 4.4 implies that There exist two $v$-critical sets, $X_1$ on $\{y, z\}$ and $X_2$ on $\{x, z\}$. If the edges $xz$ and $yz$ are both present in $E(H)$, then $\{x, y\}$ is a 2-separator, contradicting an hypothesis of the lemma. Thus we may assume, without loss of generality, that $yz \notin E$. Then for the $v$-critical set $X_1$ on $y, z$ we must have $|X_1| \geq 3$. By Lemma 2.16(b) $H[X_1]$ is 2-connected, and hence $z$ has two neighbours in $X_1$. If $z$ has no neighbours in $X_2$ then $xz \notin E(X_2)$, $|X_2| \geq 3$, and $z$ is an isolated vertex in $H[X_2]$. This would contradict Lemma 2.16(b). Hence $z$ has a neighbour in $X_2$. Since $z$ is a node and has two neighbours in $X_1$, this implies that $|X_1 \cap X_2| \geq 2$. By Lemma 4.2 this gives that $X_1 \cup X_2$ is also critical. Since $d(v, X_1 \cup X_2) \geq 3$, Lemma 2.15 implies that $X_1 \cup X_2 = V(H) - v$. Thus (a) holds.

Now suppose that $v$ is not adjacent to a node. Since $v$ is non-admissible, Lemma 4.4 implies that there exist three $v$-critical sets $X, Y, Z$ on $\{y, z\}$, $\{x, z\}$ and $\{x, y\}$, respectively. Suppose that no two of these sets intersect each other in at least two vertices. Then we also have $X \cap Y \cap Z = \emptyset$. Lemma 2.2 implies that $X \cup Y \cup Z$ is critical and $d(X, Y, Z) = 0$. Since $d(v, X \cup Y \cup Z) = 3$, we deduce that $X \cup Y \cup Z = V - v$ (otherwise $(X \cup Y \cup Z) + v$ violates Lemma 2.15). Since $|V| \geq 5$, at least one of the three critical sets $X, Y, Z$ (say, $X$) satisfies $|X| \geq 3$. But we have $d(X, Y, Z) = 0$, and hence $y, z$ is a 2-separator in $H$, contradicting an hypothesis of the lemma. This contradiction shows that we can choose two sets $X_1, X_2 \in \{X, Y, Z\}$ with $|X_1 \cap X_2| \geq 2$. Then $X_1 \cup X_2$ is critical by Lemma 4.2 and so $X_1 \cup X_2 = V - v$ follows, using Lemma 2.15 and $d(v, X \cup Y) = 3$. Thus (b) holds. •

The next lemma extends [1, Lemma 3.3].

**Lemma 4.6.** *Let $H = (V, E)$ be an $M$-circuit which is not a wheel, and let $v$ be a node. Let $N(v) = \{x, y, z\}$ and let $X$ be a $v$-critical set on $x, y$ with $d(z) \geq 4$ and $|X| \geq 3$. Suppose that either*
*(a) there is a non-admissible series node $u \in V - X - v$ with exactly one neighbour $w$ in $X$, and $w$ is a node, or*
*(b) there is a non-admissible leaf node $t \in V - X - v$.*
*Then either there is a 2-separation $(H_1, H_2)$ of $H$ with $X \subseteq V(H_1)$ or there is a node-critical set $X^*$ with $X \subset X^*$.*

**Proof:** Suppose first that (a) occurs and let $N(u) = \{w, p, q\}$. By our assumption $N(u) \cap X = \{w\}$ and $d(w) = 3$. Since $u$ is a series node, we can assume that $d(p) = 3$ and $d(q) \geq 4$. Since $u$ is non-admissible, there exists a $u$-critical set $Y$ on $\{w, p\}$ by Lemma 4.4. Now $H$ is not a wheel, and hence $H[V_3]$ contains no cycles by Lemma 4.1. Thus $pw \notin E$ and hence $|Y| \geq 3$. This implies, by Lemma 2.6(a), that $G[Y]$ is 2-connected, and hence $Y$ contains two neighbours of $w$. Since $|X| \geq 3$, Lemma 2.6(a) implies that $G[X]$ is 2-connected, and hence at least one of the neighbours of $w$ in $Y$ must be in $X$. Thus $|X \cap Y| \geq 2$. Let $X^* = X \cup Y$. We have $X^* \subseteq V - u - q$, and Lemma 4.2 implies that $X^*$ is a $u$-critical set on $\{w, p\}$. Since $d(q) \geq 4$ and $p \notin X$, the set $X^*$ is a node-critical set which properly contains $X$.

We next suppose that (b) occurs. We must have $|N(t) \cap X| \leq 2$, since $|N(t) \cap X| = 3$ would imply that $X + t$ violates Lemma 2.15(c). If $|N(t) \cap X| = 2$ then $X + t$ is also critical and by choosing $X^* = X + t$ the lemma follows. Thus we may assume that $|N(t) \cap X| \leq 1$.

Since $t$ is a non-admissible leaf node, Lemma 4.5 implies that either there is a 2-separator consisting of two neighbours of $t$ or there exist two $t$-critical sets $Y_1$ and $Y_2$ with $Y_1 \cup Y_2 = V - t$, $|Y_1 \cap Y_2| \geq 2$, and so that if $t$ has a neighbour $r$ which is a node then $r \in Y_1 \cap Y_2$. In the former case we are done (since $G[X]$ is 2-connected by Lemma 2.6(a) and hence $X$ is contained in one side of the corresponding 2-separation). Suppose that the latter case holds. Note that $Y_1$ and $Y_2$ are node-critical since $t$ is a leaf node and $|Y_1|, |Y_2| \geq 3$. Since $Y_1 \cup Y_2 = V - t$, $t \notin X$, and $|X| \geq 3$, we have $|X \cap Y_1| \geq 2$ or $|X \cap Y_2| \geq 2$. Let us assume, without loss of generality, that $|X \cap Y_1| \geq 2$ holds. By Lemma 4.2, $X \cup Y_1$ is a critical set. If $N(t) \cap X \subseteq Y_1$, then the lemma follows by choosing $X^* = X \cup Y_1$. (The set $X^*$ is $t$-critical and the unique neighbour of $t$ in $V - X^*$ has degree four in $H$.)

Thus we may assume that $N(t) \cap X = \{s\}$ and $s \notin Y_1$ holds. This implies that $d(s) \geq 4$, since if $d(s) = 3$ then we have $s \in Y_1 \cap Y_2$ as noted above. Since $Y_1 \cup Y_2 = V - t$, we have $s \in Y_2$. Hence if $|X \cap Y_2| \geq 2$ then we are done, as above, by choosing the $t$-critical set $X^* = X \cup Y_2$. Thus, we may suppose that $|X \cap Y_2| = 1$. Since $d(t, X \cup Y_1) = 3$, and $X \cup Y_1$ is critical, Lemma 2.15 implies $X \cup Y_1 = V - t$. Since $Y_1 \cup Y_2 = V - t$, we have $(X - s) \subseteq Y_1$. Thus $V - Y_1 = \{s, t\}$. This contradicts Lemma 4.3, since $d(s) \geq 4$, and completes the proof of the lemma. •

# 5 Ear decompositions and admissible splittings in $M$-connected graphs

Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid and let $C_1, C_2, ..., C_t$ be a non-empty sequence of circuits of $\mathcal{M}$. Let $D_j = C_1 \cup C_2 \cup ... \cup C_j$ for $1 \leq j \leq t$. We say that $C_1, C_2, ..., C_t$ is a *partial ear decomposition* of $\mathcal{M}$ if for every $2 \leq i \leq t$ the following properties hold:

(E1) $C_i \cap D_{i-1} \neq \emptyset$,

(E2) $C_i - D_{i-1} \neq \emptyset$,

(E3) no circuit $C_i'$ satisfying (E1) and (E2) has $C_i' - D_{i-1}$ properly contained in $C_i - D_{i-1}$.

The set $C_i - D_{i-1}$ is called the *lobe* of circuit $C_i$, and is denoted by $\tilde{C}_i$. An *ear decomposition* of $M$ is a partial ear decomposition with $D_t = E$. We need the following facts about ear decompositions. The proof of (a) and (b) in the next lemma can be found in [5]. The proof of (c) is easy and is omitted.

**Lemma 5.1.** *Let $\mathcal{M}$ be a matroid. Then*
*(a) $\mathcal{M}$ is connected if and only if $\mathcal{M}$ has an ear decomposition.*
*(b) If $\mathcal{M}$ is connected then any partial ear decomposition of $\mathcal{M}$ can be extended to an ear decomposition of $\mathcal{M}$.*
*(c) If $C_1, C_2, ..., C_t$ is an ear decomposition of $\mathcal{M}$ then*

$$r(D_i) - r(D_{i-1}) = |\tilde{C}_i| - 1 \quad \text{for} \quad 2 \leq i \leq t. \tag{6}$$

As an example, an ear-decomposition $C_1, C_2$ of the rigidity matroid of $G = K_{3,5}$ can be obtained by taking the edge sets of two different $K_{3,4}$ subgraphs of $G$. These subgraphs are (intersecting) $M$-circuits in $G$ and their union contains all edges of $G$.

**Lemma 5.2.** *Let $G = (V, E)$ be an $M$-connected graph and $H_1, H_2, \ldots, H_t$ be the $M$-circuits of $G$ induced by an ear decomposition of $\mathcal{M}(G)$ with $t \geq 2$. Let $H = H_t$, $Y = V(H) - \cup_{i=1}^{t-1} V(H_i)$, $Z = E(H) - \cup_{i=1}^{t-1} E(H_i)$, and let $X = V(H) - Y$. Then:*
*(a) either $Y = \emptyset$ and $|Z| = 1$, or $Y \neq \emptyset$ and every edge $e \in Z$ is incident to $Y$;*
*(b) $|Z| = 2|Y| + 1$;*
*(c) $X$ is critical in $H$;*
*(d) $G[Y]$ is connected.*
*(e) If $G$ is 3-connected then $|X| \geq 3$.*

**Proof:** Since $M$-connected graphs are rigid, it follows that $G$, $\cup_{i=1}^{t-1} H_i$, and $H$ are all rigid. Thus (E3) implies that (a) holds. Furthermore, $r(E) = 2|V| - 3$ and $r(\cup_{i=1}^{t-1} E(H_i)) = 2|V - Y| - 3$. By Lemma 5.1(c) this implies that $2|Y| = |\tilde{C}_t| - 1 = |Z| - 1$. This gives (b).

Since $H$ is an $M$-circuit, we have $|E(H)| = 2|V(H)| - 2$. Hence, since $|X| \geq 2$, (b) implies that $X$ is critical in $H$ and hence (c) holds.

To prove (d) suppose that $Y$ can be partitioned into two non-empty sets $Y_1, Y_2$ with $d(Y_1, Y_2) = 0$. Since $X$ is critical and $H$ is an $M$-circuit, we must have $i(Y_j) + d(Y_j, X) \leq 2|Y_j|$ for $j = 1, 2$. This gives $|Z| = \sum_{j=1}^{2} i(Y_j) + d(Y_j, X) \leq 2(|Y_1| + |Y_2|) \leq 2|Y|$, contradicting (b). Property (e) follows from the fact that either $Y \neq \emptyset$ and $X$ is a separator in $G$ (using (c)), or $Y = \emptyset$ and $|X| = |V(H)| \geq 4$ (since $H$ is an $M$-circuit). •

Let $G$ be an $M$-circuit, $v$ be a node of $G$ and $N(v) = \{x, y, z\}$. Since $G - xz$ is rigid, $G - v$ is rigid by Lemma 2.8(a). Thus $G_v^{x,y} = G - v + xy$ is rigid. Since $|V(G_v^{x,y})| = 2|E(G_v^{x,y})| - 2$, $G_v^{x,y}$ contains a unique $M$-circuit $C$. We have $C = G_v^{x,y}$ if and only if the splitting of $v$ on $vx, vy$ is admissible. If not, $V(C)$ is the minimal $v$-critical set on $\{x, y\}$ in $G$.

**Lemma 5.3.** *Let $G$ be an $M$-connected graph and $H_1, H_2, \ldots, H_t$ be the $M$-circuits of $G$ induced by an ear decomposition of $\mathcal{M}(G)$ with $t \geq 2$. Let $H = H_t$, $Y = V(H) - \cup_{i=1}^{t-1} V(H_i)$ and $X = V(H) - Y$. Let $v$ be a node of $G$ in $Y$, and let $x, y \in N(v)$ with $x \notin X$. Let $C$ be the unique $M$-circuit in $H_v^{x,y}$. If $E(C) \cap E_H(X) \neq \emptyset$ and $E(H_v^{x,y}) - E_H(X) \subset E(C)$, then splitting $v$ on $vx, vy$ is admissible in $G$.*

**Proof:** Let $H' = H_v^{x,y}$, and $N(v) = \{x, y, z\}$. It suffices to show that $E(H_1), E(H_2), \ldots, E(H_{t-1}), E(C)$ is an ear-decomposition of $\mathcal{M}(G_v^{x,y})$ since this will imply that $G_v^{x,y}$ is $M$-connected. Let $D_{t-1} = \cup_{i=1}^{t-1} E(H_i)$. Then $E_H(X) \subseteq D_{t-1}$ by Lemma 5.2(a). Since $E(H_v^{x,y}) - E_H(X) \subset E(C)$, $\cup_{i=1}^{t-1} E(H_i) \cup E(C) = E(G_v^{x,y})$. Properties (E1), (E2) and (E3) are clearly satisfied for $2 \leq i \leq t-1$. Property (E1) follows for '$i = t$' from the hypothesis that $E(C) \cap E_H(X) \neq \emptyset$ and the fact that $E_H(X) \subseteq D_{t-1}$. Property (E2) holds for '$i = t$' since $xy \in E(C) - D_{t-1}$. To see that (E3) holds for '$i = t$' we proceed by contradiction. Suppose that there is an

$M$-circuit $C'$ with $E(C') \cap D_{t-1} \neq \emptyset \neq E(C') - D_{t-1}$ and $C' - D_{t-1} \subset C - D_{t-1}$. Since $E(H_1), E(H_2), \ldots, E(H_t)$ satisfies (E3), we must have $xy \in E(C')$. Let $C''$ be obtained from $C'$ by a 1-extension, which deletes the edge $xy$, adds a new vertex $v$, and the edges $vx, vy, vz$. Now $C''$ violates (E3) with respect to the ear decomposition $E(H_1), E(H_2), \ldots, E(H_t)$ of $\mathcal{M}(G)$, a contradiction. •

Note that if splitting $v$ along $vx, vy$ is admissible in $H$, then the hypotheses of Lemma 5.3 are trivially satisfied since we have $C = H_v^{x,y}$.

**Theorem 5.4.** *Let $G$ be a 3-connected $M$-connected graph which is not an $M$-circuit. Let $H_1, H_2, \ldots, H_t$ be the $M$-circuits of $G$ induced by an ear decomposition of $\mathcal{M}(G)$. Suppose that $G - e$ is not $M$-connected for all $e \in E(H_t) - \cup_{i=1}^{t-1} E(H_i)$. Then $V(H_t) - \cup_{i=1}^{t-1} V(H_i)$ contains an admissible node of $G$.*

**Proof:** Suppose the theorem is false and let $G$ be a counterexample. Since $G$ is not an $M$-circuit, we have $t \geq 2$. Let $H = H_t$, $Y = V(H) - \cup_{i=1}^{t-1} V(H_i)$, $X = V(H) - Y$. Since $G - e$ is not $M$-connected for all $e \in E(H_t) - \cup_{i=1}^{t-1} E(H_i)$, we have $Y \neq \emptyset$, by Lemma 5.2(a). Let $D = \cup_{i=1}^{t-1} V(H_i)$. Since $G$ is 3-connected, we have $|X| \geq 3$ by Lemma 5.2(e). Note that every edge $e \in E(H) - \cup_{i=1}^{t-1} E(H_i)$ is incident to $Y$ by Lemma 5.2(a).

Suppose $H = K_4$. Then the 3-connectivity of $G$ implies that $|Y| = 1$. Let $V(H) = \{v, x, y, z\}$, where $Y = \{y\}$. Then $G - vx$ is a 1-extension of $D_{t-1}$. Thus, by Lemma 3.9, $G - vx$ is $M$-connected, contradicting an hypothesis of the theorem. Thus $H \neq K_4$.

By Lemmas 4.3 and 5.2(c), $Y$ contains a node. Since $G$ is not an $M$-circuit, $G \neq W_n$. Lemma 4.1 implies that we can choose a node $v$ of $G$ in $Y$ such that $v$ is a leaf in $G[Y \cap V_3] = H[Y \cap V_3]$. Let $N(v) = \{x, y, z\}$.

**Claim 5.5.** *$v$ does not have three neighbours in $X$.*

**Proof:** For a contradiction suppose $N(v) \subset X$. Then, by Lemma 5.2(d), we must have $|Y| = \{v\}$. If $xy \in E(D)$ then $G - xy$ is a 1-extension of $D$. Thus $G - xy$ is $M$-connected by Lemma 3.9, which contradicts an hypothesis of the theorem. Hence $xy \notin E(D)$ and splitting off $v$ on the pair $vx, vy$ gives $D + xy$, which is again $M$-connected by Lemma 3.9. Thus $v$ is an admissible node of $G$. •

**Claim 5.6.** *$v$ does not have two neighbours in $X$.*

**Proof:** Let $N(v) \cap X = \{x, y\}$. If splitting off $v$ along $xz$ or $yz$ is admissible in $H$ then by Lemma 5.3 it is an admissible split in $G$. Hence, by Lemma 4.4, we may assume that there exist two minimal critical sets $X_1, X_2$ in $H$ with $x, z \in X_1$ and $y, z \in X_2$. Note that the minimality of $X_1$ implies that the unique $M$-circuit $C'$ in $H_v^{x,z}$ satisfies $V(C') = X_1$.

Suppose $|X \cap X_1| \geq 2$. Then $X \cup X_1$ and $X \cap X_1$ are critical and $d(X, X_1) = 0$ by Lemma 4.2. Since $d(v, X \cup X_1) = 3$, Lemma 2.15 now implies that $X \cup X_1 = H - v$. Hence $(E(H_v^{x,z}) - E(X)) \subseteq E(C')$. Since $X \cap X_1$ is critical, $H[X \cap X_1]$ is connected (it is either $K_2$ or is 2-connected by Lemma 2.6(a)) and hence $E(X) \cap E(C') \neq \emptyset$.

Thus $v$ is admissible in $G$ by Lemma 5.3. Hence $X \cap X_1 = \{x\}$ and, by symmetry, $X \cap X_2 = \{y\}$.

If $|X_1 \cap X_2| \geq 2$ then $X_1 \cup X_2 = V(H) - v$ and $\{x, y\}$ is a 2-separator in $G$. This contradicts the 3-connectivity of $G$ and hence $|X_1 \cap X_2| = 1$. Now Lemma 2.2 implies that $d(X, X_1, X_2) = 0$. This again implies that $\{x, y\}$ is a 2-separator in $G$, and gives a contradiction. •

**Claim 5.7.** *There is a set $X' \subset V(H)$ such that $X'$ is node-critical in $H$ and $X \subseteq X'$.*

**Proof:** It follows from Claims 5.5, 5.6 that $v$ has at most one neighbour in $X$.

**Case 1** $v$ has exactly one neighbour, say $x$, in $X$.
Since $v$ is a leaf, we may assume without loss of generality that $d_H(y) \geq 4$. If splitting off $v$ along $xz$ or $yz$ is admissible in $H$ then by Lemma 5.3 it is an admissible split in $G$. Hence, by Lemma 4.4, we may assume that there exist two minimal critical sets $X_1, X_2$ in $H$ with $x, z \in X_1$ and $y, z \in X_2$. If $|X \cap X_1| \geq 2$ then Lemma 4.2 implies that $X \cup X_1$ is the desired node critical set containing $X$ in $H$. Hence

$$X \cap X_1 = \{x\}. \tag{7}$$

Suppose $|X \cap X_2| \geq 2$. Then Lemma 4.2 implies that $X \cup X_2$ is critical and $d(X, X_2) = 0$. Since $N(v) \subseteq X \cup X_2$, Lemma 2.15 gives $X \cup X_2 = V(H) - v$. Hence the unique circuit $C'$ in $H_v^{yz}$ satisfies $(E(H_v^{y,z}) - E(X)) \subseteq E(C')$ and $E(X) \cap E(C') \neq \emptyset$ (because $X \cap X_2$ is also critical, so $H[X \cap X_2]$ is connected). Thus $v$ is admissible in $G$ by Lemma 5.3. Hence

$$|X \cap X_2| \leq 1. \tag{8}$$

If $|X_1 \cap X_2| \geq 2$ then we may deduce as above that $X_1 \cup X_2 = V(H) - v$ must hold. Since $|X| \geq 3$, this contradicts either (7) or (8). Thus $X_1 \cap X_2 = \{z\}$. Hence $z$ is not a node by Lemma 4.5(a). Thus $d_H(z) \geq 4$. We now choose a critical set $X_3$ in $H$ with $x, y \in X_3$ (if it did not exist then splitting $v$ along $xy$ would be admissible in $G$). By symmetry we have $|X_3 \cap X_2| = 1$. If $|X_3 \cap X| \geq 2$ then $X \cup X_3$ is the desired node-critical set. Hence $|X_3 \cap X| = 1$ and Lemma 2.2 gives that $X_1 \cup X_2 \cup X_3$ is critical. Hence $X_1 \cup X_2 \cup X_3 = V(H) - v$. We may now deduce that $|X| \leq 2$, since $X \subseteq X_1 \cup X_2 \cup X_3$ and $X \cap (X_1 \cup X_3) = \{x\}$ and $|X \cap X_2| \leq 1$. This contradicts the fact that $|X| \geq 3$.

**Case 2** $N(v) \cap X = \emptyset$.
We have $x, y, z \in Y$. Since $v$ is a leaf we may assume, without loss of generality, that $d_H(x) \geq 4$ and $d_H(y) \geq 4$. Lemma 5.3 implies that $v$ is not splittable along $yz$ or $zx$. Thus there exist minimal critical sets $X_1$ and $X_2$ in $H$ on $\{y, z\}$ and $\{z, x\}$ respectively. If two neighbours of $v$ form a 2-separator in $H$, then the fact that $G[X]$ is connected implies that this will also be a 2-separator in $G$. This contradicts the 3-connectivity of $G$. Lemma 4.5 now implies that $|X_1 \cap X_2| \geq 2$ and $X_1 \cup X_2 = V(H) - v$ (possily after renaming $x, y, z$ in the case when $d_H(z) \geq 4$). Since $|X| \geq 3$, we may assume by symmetry that $|X_1 \cap X| \geq 2$. Now Lemma 4.2

implies that $X \cup X_1$ is the required $v$-critical set containing $X$. •

Choose a maximal node-critical set $X^* \subset V(H)$ with $X \subseteq X^*$. Then $X^*$ is $v$-critical for some node $v$. Applying Lemma 4.3 to the critical set $X^* \cup \{v\}$, we deduce that $H - X^* - v$ contains a node. Lemma 4.1 now implies that we may choose a leaf $w$ in $H[V_3 - X^* - v]$. Then $w$ has at most one neighbour in $X^*$ (otherwise $X^* + w$ would either contradict Lemma 2.15 or be a larger node critical set than $X^*$.) Thus $w$ is either a leaf in $H[V_3]$ or is a series node with a unique neighbour $r$ in $X^*$, such that $r$ is a node. Using Lemma 4.6, the 3-connectivity of $G$ and the maximality of $X^*$, we can deduce that $w$ is admissible in $H$ (and hence in $G$). This proves the theorem. •

We shall also need

**Theorem 5.8.** *[1, Theorem 3.8] Let $G$ be a 3-connected $M$-circuit with at least five vertices. Then either $G$ has three non-adjacent admissible nodes or $G$ has four admissible nodes.*

Theorems 5.4 and 5.8, and Lemmas 3.3 and 3.4 imply the following extension of [1, Theorem 4.4].

**Corollary 5.9.** *$G = (V, E)$ is $M$-connected if and only if $G$ is a connected graph obtained from disjoint copies of $K_4$'s by recursively applying edge additions and 1-extensions within a connected component, and taking 2-sums of different connected components.*

# 6 Bricks

A graph $G$ is a *brick* if it is 3-connected and $M$-connected. A brick $G = (V, E)$ is said to be *minimal* if $G - e$ is not a brick for all $e \in E$. An edge $f$ of $G$ is *admissible* if $G - f$ is $M$-connected. A node $v$ of $G$ is *feasible* if $G_v$ is a brick for some splitting $G_v$ of $G$ at $v$. A *fragment* in a 2-connected graph $H$ is a set $X \subseteq V(H)$ such that $|N_H(X)| = 2$ and $1 \leq |X| \leq |V(H)| - 3$. Let $N$ be a 2-separator in $H$, $x, y \in V(H)$ and $e \in E(H)$. We say that $N$ *separates* $x$ *and* $y$ if $x$ and $y$ belong to different components of $H - N$. We say that $N$ *separates* $x$ *and* $e$ if either $x$ and $e$ belong to different components of $H - N$, or $e$ is an edge from $N$ to a component of $H - N$ which does not contain $x$.

**Theorem 6.1.** *Let $G$ be a minimal brick. If $G \neq K_4$ then $G$ has a feasible node.*

**Proof:** We proceed by contradiction. Suppose the theorem is false and let $G$ be a counterexample with as few vertices as possible. If $G$ is minimally $M$-connected then $G$ has an admissible splitting $G_w^{x,y}$ by Theorems 5.4 and 5.8. Since $G$ is a counterexample to the theorem, $G' = G_w^{x,y}$ is not 3-connected. On the other hand, if $G$ is not minimally $M$-connected, then $G$ has an admissible edge $f$. Since $G$ is a minimal brick, $G' = G - f$ is not 3-connected. We now consider all possible choices for an admissible splitting and an admissible edge, and choose one such that some fragment $X$ of the resulting $M$-connected graph $G'$ is minimal with respect to inclusion.

We shall prove that $X$ contains a feasible node of $G$. Since $G'$ is $M$-connected, $G'$ has minimum degree at least three and hence $|X| \geq 2$. By Lemma 3.5, $G'$ has no polygons. Let $N := N_{G'}(X) = \{u, v\}$. Let $H, K$ be the cleavage graphs obtained by cleaving $G'$ at $\{u, v\}$, where $X = V(H) - \{u, v\}$. Note that the minimality of $X$ and the fact that $G'$ has no polygons imply that $H$ is a cleavage unit of $G'$, and the 3-connectivity of $G$ implies that $K - \{u, v\}$ is connected.

If $G' = G_w^{x,y}$ and $N(w) = \{x, y, z\}$, then let $V^*(H) = X - \{x, y, z\}$ and $E^*(H) = (E(H) \cap E(G)) - uv$. (The 3-connectivity of $G$ implies that either $x, y \in X \cup N$ and $z \in V(K) - N$, or $x, y \in V(K)$ and $z \in X$.) On the other hand, if $G' = G - f$ and $f = yz$, then let $V^*(H) = X - \{y, z\}$ and $E^*(H) = E(H) - uv$. (The 3-connectivity of $G$ implies that $\{y, z\} \cap X \neq \emptyset$ and $\{y, z\} \cap (V(K) - N) \neq \emptyset$.) Let $\theta = xy$ if $G' = G_w^{x,y}$ and $xy \in E(H)$, let $\theta = z$ if $G' = G_w^{x,y}$ and $xy \notin E(H)$, and let $\theta$ be the unique vertex of $X$ which is incident to $f$ in $G$ if $G' = G - f$.

**Claim 6.2.** *$H$ is 3-connected.*

**Proof:** This follows since $G'$ has no polygons and hence all its cleavage units are 3-connected. •

**Claim 6.3.** *$uv \notin E(G)$.*

**Proof:** Suppose $uv \in E(G)$. Since $G'$ is $M$-connected, and $u, v$ is a 2-separator, Lemma 3.8 implies that $G' - uv$ is $M$-connected. Since $G - uv$ is obtained from $G' - uv$ by either an edge addition or a 1-extension, $G - uv$ is $M$-connected by Lemma 3.9. Futhermore, $G' - uv$ contains three internally disjoint $uv$-paths (two in $H - uv$ by Claim 6.2 and one in $K - uv$). Thus $G - uv$ has three internally disjoint $uv$-paths and the 3-connectivity of $G$ implies that $G - uv$ is 3-connected. This contradicts the fact that $G$ is a minimal brick. •

**Claim 6.4.** *$H$ and $K$ are $M$-connected.*

**Proof:** This follows from Lemma 3.4 since $G'$ is $M$-connected and $G' = H \oplus_2 K$. •

**Claim 6.5.** *Suppose that $G - e$ is $M$-connected for some $e \in E^*(H)$. Then $H - \{u, v, e\}$ is connected.*

**Proof:** Suppose $H - \{u, v, e\}$ has two components $H_1, H_2$. Choose $i \in \{1, 2\}$ such that $\theta \notin V(H_i) \cup E(H_i)$. Then $V(H_i)$ is a fragment of $G - e$ which is properly contained in $X$. This contradicts the choice of $G'$ and $X$. •

**Claim 6.6.** *$G - e$ is not $M$-connected for all $e \in E^*(H)$.*

**Proof:** Suppose that $G - e$ is $M$-connected for some edge $e = ab \in E^*(H)$. Since $G$ is a minimal brick, $G - e$ is not 3-connected. Let $L$ be a 2-separator in $G - e$. Since

$G$ is 3-connected, $L$ separates $a$ and $b$. If $G' = G_w^{x,y}$ then Lemma 2.20 implies that $w \notin L$.

Since $G'$ is $M$-connected, it is 2-rigid. Hence the graph $G'' = G' - e$ is rigid. Thus $G''$ is 2-connected by Lemma 2.6(a). Clearly, $L$ and $N$ are 2-separators in $G''$. By Lemma 3.6, $L$ and $N$ do not cross in $G''$. By Claim 6.5, $H - \{u, v, e\}$ is connected. Since $a, b \in X \cup N$ and $L$ separates $a$ and $b$ in $G - e$, we have $L \cap X \neq \emptyset$ and $G''[X]$ is a component of $G'' - N$. Since $L$ and $N$ do not cross, we have $L \cap (V - X - N) = \emptyset$. Since $K - N$ is connected, some component $J'$ of $G'' - L = G' - e - L$ contains $V - X - N$. Let $J$ be the component of $G - e - L$ which contains $V - X - N$. Then $V - X \subset V(J) \cup N_{G-e}(J)$. Moreover, if $G' = G_w$, then the neighbour(s) of $w$ in $X$ are contained in $V(J) \cup N_{G-e}(J)$, and, if $G' = G - f$ then the endvertex of $f$ in $X$ is contained in $V(J) \cup N_{G-e}(J)$. This implies in both cases that the vertex set of the component of $G - e - L$ distinct from $J$ is a proper subset of $X$. This contradicts the minimality of $X$. •

**Claim 6.7.** *$H - e$ is not $M$-connected for all $e \in E^*(H)$.*

**Proof:** Suppose $H - e$ is $M$-connected. Then $G' - e = (H - e) \oplus_2 K$ and $G' - e$ is $M$-connected by Claim 6.4 and Lemma 3.3. Since, by Lemma 3.9, the property of being $M$-connected is preserved by edge addition and 1-extension, it follows that $G - e$ is $M$-connected. This contradicts Claim 6.6. •

**Claim 6.8.** *Suppose $p \in V^*(H)$ is a node of $G$, $N_G(p) = \{q, s, t\}$, and $G_p^{s,t}$ is $M$-connected. Then $H_p^{s,t} - \{u, v\}$ is connected.*

**Proof:** Suppose $H_p^{s,t} - \{u, v\}$ is disconnected. Then $H - \{u, v\}$ has a 1-separation $(H_1, H_2)$ where $V(H_1) \cap V(H_2) = \{p\}$, $s, t \in V(H_1)$ and $q \in V(H_2)$. Choose $i \in \{1, 2\}$ such that $\theta \notin V(H_i) \cup E(H_i)$. Then $V(H_i) - p$ is a fragment of $G_p^{s,t}$ which is properly contained in $X$. This contradicts the choice of $G'$ and $X$. •

**Claim 6.9.** *$G_p$ is not $M$-connected for all nodes $p$ of $G$ in $V^*(H)$.*

**Proof:** Suppose that $G_p = G_p^{s,t}$ is $M$-connected for some node $p$ of $G$ in $V^*(H)$, with $N_G(p) = \{q, s, t\}$. Since $G$ is a counterexample to the theorem, $G_p^{s,t}$ is not 3-connected. Let $L$ be a 2-separator in $G_p^{s,t}$. Since $G$ is 3-connected, $L$ separates $st$ and $q$. If $G' = G_w^{x,y}$ then Lemma 2.20 implies that $w \notin L$.

Since $G'$ is $M$-connected, it is 2-rigid. Hence $G' - pq$ is rigid. Since $G' - p$ is obtained from $G' - pq$ by deleting a vertex of degree two, it is rigid by Lemma 2.8(b). Since $G'' = (G')_p^{s,t}$ is obtained from $G' - p$ by an edge addition, it is also rigid. Thus $G''$ is 2-connected by Lemma 2.6(a). Clearly, $L$ and $N$ are 2-separators in $G''$. By Lemma 3.6, $L$ and $N$ do not cross in $G''$. By Claim 6.8, $H_p^{s,t} - \{u, v\}$ is connected. Since $q, s, t \in X \cup N$ and $L$ separates $st$ and $q$ in $G''$, we have $L \cap X \neq \emptyset$ and $G''[X - p]$ is a component of $G'' - N$. Since $L$ and $N$ do not cross, we must have $L \cap (V - X - N) = \emptyset$. Hence some component $J'$ of $G'' - L = G'_p - L$ contains $V - X - N$. Let $J$ be the component of $G_p^{s,t} - L$ which contains $V - X - N$. Thus $V - X \subset V(J) \cup N_{G_p}(J)$. Moreover, if $G' = G_w$, then the neighbour(s) of $w$ in $X$ are also contained in $V(J) \cup N_{G_p}(J)$, and, if $G' = G - f$ then the endvertex of $f$ in $X$ is contained in $V(J) \cup N_{G_p}(J)$. This implies in both cases that the vertex set of the component of $G_p - L$ which is distinct from $J$ is a proper subset of $X$. This contradicts the minimality of $X$. •

**Claim 6.10.** *$H_p$ is not $M$-connected for all nodes $p$ of $G$ in $V^*(H)$.*

**Proof:** Suppose $H_p$ is $M$-connected. Then $G'_p = (H_p) \oplus_2 K$ and $G'_p$ is $M$-connected by Claim 6.4 and Lemma 3.3. Since the property of being $M$-connected is preserved by edge addition and 1-extension, it follows that $G_p$ is $M$-connected. This contradicts Claim 6.9. •

**Claim 6.11.** *$H$ is an $M$-circuit.*

**Proof:** Suppose $H$ is not an $M$-circuit. Since $H$ is $M$-connected by Claim 6.4, we may choose an ear decomposition $C_1, C_2, \ldots, C_t$ of $\mathcal{M}(G)$. Let $H_i$ be the $M$-circuit of $H$ induced by $C_i$ for $1 \leq i \leq t$. By Lemma 5.1(b) we may suppose that the ear decomposition has been chosen such that $H_1$ contains $uv$ and $\theta$. By Claims 6.2, 6.7 and Theorem 5.4, $H_t - \cup_{i=1}^{t-1} H_i$ contains an admissible node $p$ of $G$ in $V^*(H)$. This contradicts Claim 6.10. •

**Claim 6.12.** *$H$ is isomorphic to $K_4$.*

**Proof:** Suppose $H$ is not isomorphic to $K_4$. By Claim 6.10, no node of $H$ in $V^*(H)$ is admissible in $H$. Since $uv \in E(H)$, Claim 6.2 and Theorem 5.8 imply that $G' = G_w^{x,y}$, $x, y \in V(H)$, and $u, v, x, y$ are the only admissible nodes in $H$. We shall show that $x$ is a feasible node in $G$.

Since $x$ is an admissible node of $H$, $H_x^{s,t}$ is $M$-connected for some $s, t \in N_H(x)$. Let $N_H(x) = \{q, s, t\}$. Since $xy$ is an edge of $H$ and $y$ is a node of $H$, we must have $y \in \{s, t\}$. Without loss of generality, $y = t$. Since $(G')_x^{s,y} = H_x^{s,y} \oplus_2 K$, Claim 6.4 and Lemma 3.3 imply that $(G')_x^{s,y}$ is $M$-connected. Since $G_x^{s,w}$ is a 1-extension of $(G')_x^{s,y}$ and since the property of being $M$-connected is preserved by 1-extension (by Lemma 3.9), it follows that $G_x^{s,w}$ is $M$-connected.

Suppose $H_x^{s,y} - \{u, v\}$ is disconnected. Then $H - \{u, v\}$ has a 1-separation $(H_1, H_2)$ where $V(H_1) \cap V(H_2) = \{x\}$, $s, y \in V(H_1)$ and $q \in V(H_2)$. Then $V(H_2) - x$ is a fragment of $G_x^{s,w}$ which is properly contained in $X$. This contradicts the choice of $G'$ and $X$. Thus $H_x^{s,y} - \{u, v\}$ is connected.

Since $G$ is a counterexample to the theorem, $G_x^{s,w}$ is not 3-connected. Let $L$ be a 2-separator in $G_x^{s,w}$. Since $G$ is 3-connected, $L$ separates $sw$ and $q$. Since $G'$ is $M$-connected, it is 2-rigid. Hence $G' - xq$ is rigid. Since $G' - x$ is obtained from $G' - xq$ by deleting a vertex of degree two, it is rigid by Lemma 2.8(b). Since

$G'' = (G')_x^{sy}$ is obtained from $G' - x$ by an edge addition, it is also rigid. Thus $G''$ is 2-connected by Lemma 2.6(a). Clearly, $L$ and $N$ are 2-separators in $G''$ and $L$ separates $sy$ and $q$ in $G''$. By Lemma 3.6, $L$ and $N$ do not cross. Since $H_x^{s,y} - \{u, v\}$ is connected, $q, s, y \in X \cup N$, and $L$ separates $sy$ and $q$ in $G''$, we have $L \cap X \neq \emptyset$ and $G''[X - x]$ is a component of $G'' - N$. Since $L$ and $N$ do not cross, we must have $L \cap (V - X - N) = \emptyset$. Hence some component $J'$ of $G'' - L = (G')_x^{s,y} - L$ contains $V - X - N$. Let $J$ be the component of $G_x^{s,y} - L$ which contains $V - X - N$. Then $V - X \subset V(J) \cup N_{G_x^{s,y}}(J)$. Moreover, $w$ and $y$ are also contained in $V(J) \cup N_{G_x^{s,y}}(J)$. This implies that the vertex set of the component of $G_x^{s,y} - L$ which is distinct from $J$ is a proper subset of $X$. This contradicts the minimality of $X$. •

**Claim 6.13.** *$G' = G_w^{x,y}$, $x, y \in V(H)$, and hence $\theta = xy \in E(H)$.*

**Proof:** Suppose that the claim is false. Then $\theta$ is a vertex in $X$, and $V(H) = \{u, v, \theta, t\}$. Let $t$ be the vertex of $X$ distinct from $\theta$. Then $t$ is a node of $G$. We shall show that $G_t^{u,v}$ is a brick. Note that $uv \notin E(G)$ by Claim 6.3. Note further that $G_t^{u,v}$ can be obtained from $K$ by a sequence of either one 1-extension and one edge-addition (if $G' = G - f$), or two 1-extensions and one edge-addition (if $G' = G_w^{x,y}$). Since $K$ is $M$-connected by Claim 6.4, it follows from Lemma 3.9 that $G_t^{u,v}$ is $M$-connected. Since $\theta$ is adjacent to $u$ and $v$, there is no 2-separation separating $\theta$ from $uv$ in $G_t^{u,v}$. Thus $G_t^{u,v}$ is 3-connected and hence is a brick. •

**Claim 6.14.** *$X \neq \{x, y\}$.*

**Proof:** Suppose that $X = \{x, y\}$. Then $x, y$ are nodes of $G$. We shall show that $G_x^{w,v}$ is a brick. Note that $wv \notin E(G)$ since the neighbour of $w$ distinct from $x, y$ belongs to $V - X - N$. Note further that $G_x^{w,v}$ can be obtained from $K$ by a sequence of two 1-extensions. Since $K$ is $M$-connected by Claim 6.4, it follows from Lemma 3.9 that $G_x^{w,v}$ is $M$-connected. Suppose that $G_x^{w,v}$ is not 3-connected. Then there is a 2-separator $L$ in $G_x^{w,v}$, separating $u$ and $wv$. Since $u, w$, and $v$ are all neighbours of $y$ in $G_x^{w,v}$, we must have $y \in L$. Since $G_x^{w,v}$ is $M$-connected and $y$ is a node in $G_x^{w,v}$, this contradicts Lemma 2.20. Thus $G_x^{w,v}$ is 3-connected and hence is a brick. •

We can now complete the proof of the theorem. Using Claims 6.13 and 6.14, and relabelling if necessary, we may suppose that $X = \{x, t\}$ and $N = \{u, y\}$. Thus $x$ is a node of $G$. We shall show that $G_x^{w,t}$ is a brick. Note that $wt \notin E(G)$ since the neighbour of $w$ distinct from $x, y$ belongs to $V - X - N$. Note further that $G_x^{w,t}$ can be obtained from $K$ by a sequence of two 1-extensions. Since $K$ is $M$-connected by Claim 6.4, it follows from Lemma 3.9 that $G_x^{w,t}$ is $M$-connected. Suppose that $G_x^{w,t}$ is not 3-connected. Then there is a 2-separator $L$ in $G_x^{w,t}$, separating $u$ and $wt$. Since $ut$ is an edge of $G_x^{w,t}$, we must have $t \in L$. Since $G_x^{w,t}$ is $M$-connected and $t$ is a node in $G_x^{w,t}$, this contradicts Lemma 2.20. Thus $G_x^{w,t}$ is 3-connected and hence is a brick. •

We have the following corollaries:

**Theorem 6.15.** *$G = (V, E)$ is a brick if and only if $G$ can be obtained from $K_4$ by 1-extensions and edge additions.*

**Proof:** Since $K_4$ is $M$-connected, sufficiency follows from Lemma 3.9, and the fact that edge addition and 1-extension preserve 3-connectivity. Necessity follows easily by induction on $|E|$, using Theorem 6.1. •

Note that a brick $G$ does not necessarily have a spanning subgraph which is a 3-connected $M$-circuit. In fact, the brick $K_{3,5}$ has no spanning $M$-circuits at all since all of its $M$-circuits are isomorphic to $K_{3,4}$. This shows that one may need to alternate between the two operations of Theorem 6.15 while building up a brick from $K_4$.

Theorem 3.2 implies that a graph is a brick if and only if it is 2-rigid and 3-connected. Thus Theorem 6.15 gives an inductive construction for 3-connected 2-rigid graphs. Since $K_4$ is globally rigid, and edge addition as well as 1-extension preserve global rigidity (by the result of Connelly [4] mentioned in Section 1), we can now deduce that Hendrickson's conjecture is true in dimension 2.

**Theorem 6.16.** *Let $G$ be a graph with $|V(G)| \geq 4$. Then $G$ is globally rigid in $\mathbb{R}^2$ if and only if $G$ is 3-connected and 2-rigid.*

Thus global rigidity of frameworks is a generic property in $\mathbb{R}^2$.

Lovász and Yemini [12] proved that 6-connected graphs are 2-rigid (and that this bound is best possible). With this result and Theorem 6.16 we can show that sufficiently highly connected graphs are globally rigid. In fact, the same degree of connectivity suffices.

**Theorem 6.17.** *Let $G$ be 6-connected. Then $G$ is globally rigid in $\mathbb{R}^2$.*

This solves [6, Open question 4.47].

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